Some notes from class

2018-02-02

Pasch's Theorem (Moritz Pasch, 1882)

Theorem

Suppose a line ℓ intersects $\triangle PQR$ at a point S such that $P - S - Q$. Then ℓ intersects $\overline{PR}$ or $\overline{RQ}$.

Proof.

Pasch's Theorem (Moritz Pasch, 1882)

Theorem

Suppose a line ℓ intersects $\triangle PQR$ at a point S such that $P - S - Q$. Then ℓ intersects $\overline{PR}$ or $\overline{RQ}$.

Proof. Let H_1 and H_2 be the half-planes determined by ℓ. Since $\overline{PQ}$ intersects ℓ (at point S), the Plane Separation Axiom implies that P and Q are in opposite half-planes, and it is no loss to write $P \in H_1$ and $Q \in H_2$.

Note that either $R \in \ell$, $R \in H_1$, or $R \in H_2$. If $R \in \ell$, then ℓ intersects both $\overline{PR}$ and $\overline{RQ}$, so we are done. If $R \in H_1$, then the Plane Separation Axiom implies that $\overline{QR}$ intersects ℓ since $Q \in H_2$. Similarly, if $R \in H_2$, then $\overline{PR}$ intersects ℓ.

Soon we will prove...

...that the base angles of an isosceles triangle are congruent.

Start of argument. Let $\triangle ABC$ be a triangle with side $\overline{AB}$ congruent to side $\overline{AC}$. Draw the angle bisector of $\angle A$ and let D be the point at which it meets side $\overline{BC}$ ...

Soon we will prove...

...that the base angles of an isosceles triangle are congruent.

Start of argument. Let $\triangle ABC$ be a triangle with side $\overline{AB}$ congruent to side $\overline{AC}$. Draw the angle bisector of $\angle A$ and let D be the point at which it meets side $\overline{BC}$ ...

Theorem (Crossbar Theorem)

If X is a point in the interior of $\triangle ABC$, then $\overrightarrow{AX}$ intersects $\overline{BC}$ in a point Y such that $B - Y - C$.

Definition

If $\triangle ABC$ is a triangle, we define the *interior* of $\triangle ABC$ to be the collection of all points not on $\overline{AB}$ or $\overline{BC}$ or $\overline{CA}$ that belong to the intersection of the interiors of angles $\angle A$, $\angle B$, and $\angle C$.

Crossbar Theorem

Theorem (Crossbar Theorem)

If X is a point in the interior of $\triangle ABC$, then $\overrightarrow{AX}$ intersects $\overline{BC}$ in a point Y such that $B - Y - C$.

Proof. Since X is in the interior of $\triangle ABC$, there exist points $W \in \overrightarrow{AC}$ and $Z \in \overrightarrow{AB}$ with $X \in \overline{WZ}$ (and $X \neq W, Z$).

We claim that $\overline{WC}$ does not intersect $\overleftrightarrow{AX}$. If $\overleftrightarrow{AX}$ intersects $\overline{WC}$, then by Axiom 1, $\overleftrightarrow{AX} = \overleftrightarrow{WC}$. This forces X to be on $\overrightarrow{AC}$, which then implies the contradiction that $X = W$. Similar reasoning shows that $\overline{BZ}$ does not intersect $\overleftrightarrow{AX}$.

What now???

Isosceles Triangle Theorem

Theorem (Isosceles Triangle Theorem)

If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

Proof. Let $\triangle ABC$ be isosceles, and suppose that $AB = AC$. Then since $AB = AC$, $\angle A \cong \angle A$, and $AC = AB$, $\ldots$

Perpendicular bisectors

Theorem (Perpendicular Bisector Theorem)

A point is on the perpendicular bisector of a line segment if and only if it is equidistant from the endpoints of the line segment.

Proof. Let A and B be distinct points.

We first prove that if a point P is on the perpendicular bisector of $\overline{AB}$, then $PA = PB$.

We next show that if a point P satisfies $PA = PB$, then P is on the perpendicular bisector of $\overline{AB}$.