Some notes from class

2018-01-17

Incidence Geometry

Undefined terms: point, line, on

Axioms:

1. For each two distinct points, there is a unique line that is on both of them
2. For every line, there exist at least two distinct points that are on it.
3. There exist at least three distinct points.
4. Not all points lie on the same line.

A couple definitions

Def. If two distinct lines are on the same point, then they are said to *intersect*.

Def. Two lines that do not intersect are called *parallel* lines.

Question. Do parallel lines always exist in incidence geometry?

Theorem 1

Theorem

In incidence geometry, if two lines intersect, then the intersection is a single point.

Proof.

Theorem 2

Theorem

In incidence geometry, each point is on at least 2 lines.

Proof. Let P be a point. Then by Axiom 3, there exist distinct points Q and R with $Q, R \neq P$. By Axiom 1, there is a line ℓ on P and Q, and there is a line m on P and R. If $\ell \neq m$, then we are done because there are two lines on P. If $\ell = m$, then P, Q, and R are all on ℓ, so by Axiom 4, there is some point S not on ℓ. Now by Axiom 1, there is a line n on S and P. If $n = \ell$, this would force S to be on ℓ, which is not the case. Thus $n \neq \ell$, and we see that n and ℓ are two lines on P.

Parallel possibilities

How do parallel lines behave (in a given type of geometry)?

If ℓ is a line and P is a point not on ℓ, then

1. . . . there does not exist a line on P parallel to ℓ.
2. . . . there exists exactly one line on P parallel to ℓ.
3. . . . there exists more than one line on P parallel to ℓ.