- Suppose a_k is a sequence. The meaning of $\sum_{k=1}^{\infty} a_k$ is as follows. Define the sequence s_n of partial sums by $s_n = \sum_{k=1}^{n} a_k$. Then $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$. Thus $\sum_{k=1}^{\infty} a_k$ is nothing more than the limit of the sequence of partial sums.
- When contemplating the convergence of $\sum_{k=1}^{\infty} a_k$, it is *totally irrelevant* what the first few terms look like. In fact, it is irrelevant what the first thousand terms look like. Only the long-term behavior of a_k matters.
- It is possible that $\lim_{k\to\infty} a_k = 0$ and yet the series $\sum_{k=1}^{\infty} a_k$ may still diverge.

- **Test for divergence**: If the terms do not go to 0, then the series $\sum_n a_n$ must diverge. i.e. If $\sum_{k=1}^{\infty} a_k$ converges, then the things being added up better be small.
- Geometric series:
  - $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$
  - $a + ar + ar^2 + \cdots = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

  Do you know how to evaluate a (finite) series like $3\left(\frac{2}{5}\right)^8 + 3\left(\frac{2}{5}\right)^9 + \cdots + 3\left(\frac{2}{5}\right)^{27}$?
  What about the infinite series $3\left(\frac{2}{5}\right)^8 + 3\left(\frac{2}{5}\right)^9 + \cdots$?
- **Integral Test**: The convergence of $\sum_n a_n$ is equivalent to the convergence of a certain improper integral. Mostly, this was useful to establish that $\sum_n \frac{1}{n^p}$ converges if and only if $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges (i.e. if $p > 1$). The integral test is occasionally useful for other types of series like $\sum_n \frac{1}{n\cdot\ln(n)}$ since $\int \frac{1}{x\ln x}\,dx$ is an antiderivative that is not too hard.
- **Comparison Test** (for series): This allows us to say things like the following. Since $\sum_n \frac{1}{n^2}$ converges and $\frac{1}{n^2+1} \le \frac{1}{n^2}$, then $\sum_n \frac{1}{n^2+1}$ converges. Similarly, since $\sum_n \frac{1}{n}$ diverges and $\frac{1}{\ln(n)} \ge \frac{1}{n}$, it must be that $\sum_n \frac{1}{\ln(n)}$ diverges.
- **Limit Comparison Test:** If we see something like $\sum_n \frac{n^3+6n^2-2n+1}{n^5+7n^3-n^2-1}$, then we can formalize the idea that it "looks a lot like" $\sum_n \frac{1}{n^2}$. Let $a_n = \frac{n^3+6n^2-2n+1}{n^5+7n^3-n^2-1}$ and $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (or any other finite nonzero real number), and this tells us that the convergence of $\sum_n a_n$ is equivalent to the convergence of $\sum_n b_n$.
- **Alternating Series Test:** If the operations are alternately adding and subtracting, and if the terms (when you ignore the sign) decrease to 0, then the series must converge.
- **Ratio Test** (absolute convergence): Look at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. (Here, it matters whether the limit is bigger than 1 or less than 1.) This works great for formulas that involve factorials and some (but not all) things that involve an nth power. One fun example of a series where the Ratio Test works is $\sum_n \frac{n^9 3^n}{\sqrt{n}2^{2n}}$. The Ratio Test is not easy to use when the formula contains n^n.
- **Root Test** (absolute convergence): Look at $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. (Here, it matters whether the limit is bigger than 1 or less than 1.) This works poorly for formulas with factorials, but well for things like 3^n or n^n. e.g. It's a great test for series like $\sum_n \frac{99^n}{n^n}$.