Some notes from class

2018-03-30

Reminders

Geometric series: $a + ar + ar^2 + ar^3 + \cdots = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \longleftrightarrow \quad \int_1^{\infty} \frac{1}{x^3}\, dx$

Comparison Test: Given $\sum_n a_n$, compare the terms of $\sum_n a_n$ to the terms of some other series you kow about.

Limit Comparison Test: Given $\sum_n a_n$, look at $\lim_{n\to\infty} \frac{a_n}{b_n}$, where $\sum_n b_n$ is some series you understand.

Alternating Series Test: Are the terms decreasing to 0, and are signs alternating?

Ratio Test: Look at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. $L > 1$, $L < 1$, $L = 1$?

AST and Ratio Test

Theorem (Alternating Series Test)

The series $\sum_{n=1}^{\infty}(-1)^n a_n$ *is guaranteed to converge if:*

- $a_n \geq 0$ *for all* n, • $a_{n+1} \leq a_n$ *for all* n, • $\lim_{n\to\infty} a_n = 0$.

Theorem (Ratio Test)

Suppose $\sum_n a_n$ *is a series.*

- *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum_n a_n$ *is absolutely convergent.*
- *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, *then the series* $\sum_n a_n$ *is divergent.*
- *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, *then the Ratio Test is inconclusive.*

Examples

- $\sum_{n=0}^{\infty} 6 \left(\frac{3}{4}\right)^n$
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$
- $\sum_{n=1}^{\infty} \frac{n^{40}}{3^n}$
- $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

Not super well suited for Ratio Test

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

Not super well suited for Ratio Test

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \frac{1}{(n+1)} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \frac{1}{(n+1)} \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{-n} \frac{1}{(n+1)} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^{-n} \frac{1}{(n+1)}
\end{aligned}$$

The Root Test

Theorem (Root Test)

Suppose $\sum_n a_n$ *is a series.*

- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum_n a_n$ *is absolutely convergent.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$*, then the series* $\sum_n a_n$ *is divergent.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, then the Ratio Test is inconclusive.*

Examples

- $$\sum_{n=0}^{\infty} 5\left(\frac{3}{7}\right)^n$$
- $$\sum_{n=1}^{\infty} \left(\frac{3n+5}{4n-1}\right)^n$$
- $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
- $$\sum_{n=1}^{\infty} \frac{1}{n^n}$$