Some notes from class

2018-03-28

Reminders

Geometric series: $a + ar + ar^2 + ar^3 + \cdots = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

Harmonic series: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \text{diverges}$

Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \longleftrightarrow \quad \int_1^{\infty} \frac{1}{x^3}\, dx$

Comparison Test: Given $\sum_n a_n$, compare the terms of $\sum_n a_n$ to the terms of some other series you kow about.

Limit Comparison Test: Given $\sum_n a_n$, look at $\lim_{n\to\infty} \frac{a_n}{b_n}$, where $\sum_n b_n$ is some series you understand.

Alternating Series Test

Theorem (Alternating Series Test)

The series $\sum_{n=1}^{\infty}(-1)^n a_n$ *is guaranteed to converge if:*

- $a_n \geq 0$ *for all* n,
- $a_{n+1} \leq a_n$ *for all* n,
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Basically: Does the given series remind us of something like:

$\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt[3]{n}}$? Note that $\frac{1}{\sqrt[3]{n}}$ is decreasing and approaches 0.

Absolute Convergence

Convergent but not absolutely convergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(n)}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Absolutely convergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \sum_{n=3}^{\infty} \frac{(-1)^n}{n^2 \ln(n)}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}, \quad \sum_{n=1}^{\infty} \frac{5}{(n+1)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n!},$$

Absolute Convergence

Definition

A series $\sum_n a_n$ is said to be *absolutely convergent* if $\sum_n |a_n|$ converges.

If $\sum_n a_n$ is convergent, but not absolutely convergent, then we say $\sum_n a_n$ is *conditionally convergent*.

The Ratio Test

Theorem (Ratio Test)

Suppose $\sum_n a_n$ *is a series.*

- *If* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, *then the series* $\sum_n a_n$ *is absolutely convergent.*
- *If* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, *then the series* $\sum_n a_n$ *is divergent.*
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Basically: If $\sum_n a_n$ behaves like a geometric series $\displaystyle\sum_n br^n$ with $|r| < 1$, then $\sum_n a_n$ converges.