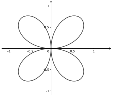
Some notes from class

2018-03-12

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta$$



Ex. $r = \sin(2\theta)$ Area of "petal" in first quadrant:

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{\sin^2(2\theta)}{2}\, d\theta \\ &= \int_0^{\pi/2} \frac{1-\cos(4\theta)}{4}\, d\theta \\ &= \frac{1}{4}\int_0^{\pi/2} 1-\cos(4\theta)\, d\theta \\ &= \frac{1}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\pi/2} \end{aligned}$$

Slope and arc length in polar coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

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Key Idea:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = f(\theta) \quad \text{a function that depends on } \theta$$

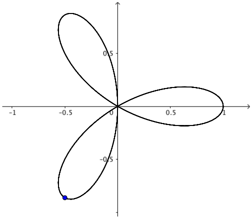
Therefore x and y both depend on the *parameter* θ.

Slope example

$$r = \cos(3\theta)$$

Find the slope when $\theta = \frac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$



$$\frac{dy}{dx} = \frac{-3\sin(3\theta)\sin\theta + \cos(3\theta)\cos\theta}{-3\sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta}$$

Arc length in polar coordinates

Recall that $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Recall $\quad x = r\cos\theta \qquad y = r\sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(r'\cos\theta - r\sin\theta\right)^2 + \left(r'\sin\theta + r\cos\theta\right)^2$$

Arc length in polar coordinates

Recall that $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Recall $\qquad x = r\cos\theta \qquad y = r\sin\theta$

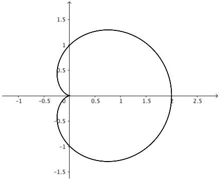
$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(r'\cos\theta - r\sin\theta\right)^2 + \left(r'\sin\theta + r\cos\theta\right)^2 \\
&= (r'\cos\theta)^2 - 2r'r\cos\theta\sin\theta + (r\sin\theta)^2 \\
&\quad +(r'\sin\theta)^2 + 2r'r\cos\theta\sin\theta + (r\cos\theta)^2 \\
&= (r')^2(\cos^2\theta + \sin^2\theta) + r^2(\cos^2\theta + \sin^2\theta) \\
&= \left(\frac{dr}{d\theta}\right)^2 + r^2
\end{aligned}$$

Arc length example 1

What is the length of the curve $r = 3$?

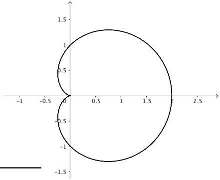
Arc length example 2

Length of "top part" of $r = 1 + \cos\theta$



Arc length example 2

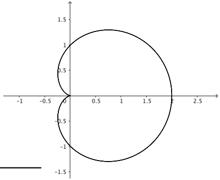
Length of "top part" of $r = 1 + \cos\theta$



$$\begin{aligned}\int_0^\pi \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta &= \int_0^\pi \sqrt{\sin^2\theta + (1+\cos\theta)^2}\, d\theta \\ &= \int_0^\pi \sqrt{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}\, d\theta \\ &= \int_0^\pi \sqrt{2 + 2\cos\theta}\, d\theta\end{aligned}$$

Arc length example 2

Length of "top part" of $r = 1 + \cos\theta$



$$\begin{aligned}
\int_0^\pi \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\; d\theta &= \int_0^\pi \sqrt{\sin^2\theta + (1+\cos\theta)^2}\, d\theta \\
&= \int_0^\pi \sqrt{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}\, d\theta \\
&= \int_0^\pi \sqrt{2 + 2\cos\theta}\, d\theta
\end{aligned}$$

Notice that $2 + 2\cos\theta = 2\frac{(1+\cos\theta)}{2} = 2\cos^2\left(\frac{\theta}{2}\right)$

Back to area

Consider the polar curves $r = 3\sin\theta$ and $r = 1 + \sin\theta$.
Find the shaded area ("between" the curves).