Some notes from class

2018-01-22

L'Hospital's Rule

Theorem (L'Hospital's Rule)

Suppose f and g are differentiable in an open interval containing $x = a$, and suppose that $g(x) \neq 0$, except possibly at $x = a$. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ (or if $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

L'Hospital's Rule

Theorem (L'Hospital's Rule)

Suppose f and g are differentiable in an open interval containing $x = a$, and suppose that $g(x) \neq 0$, except possibly at $x = a$. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ (or if $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

*(**Note:** This also works for one-sided limits, and it even works if $a = \infty$ or $a = -\infty$.)*

Roughly, why does L'Hospital's Rule work?

Rough Idea: Suppose $a = 3$ and $\lim_{x\to 3} \frac{f(x)}{g(x)}$ has $\frac{0}{0}$ form. Also suppose that $\lim_{x\to 3} \frac{f'(x)}{g'(x)} = \frac{f'(3)}{g'(3)}$. Then ...

1. The tangent line to f at $x = 3$ is $L(x) = f(3) + f'(3)(x-3)$.
2. The tangent line to g at $x = 3$ is $M(x) = g(3) + g'(3)(x-3)$.
3. $f(3) = 0$ and $g(3) = 0$ (since f and g are continuous).
4. f and g are approximately equal to their tangent lines.

$$\lim_{x\to 3} \frac{f(x)}{g(x)} = \lim_{x\to 3} \frac{f(3) + f'(3)(x-3)}{g(3) + g'(3)(x-3)} = \lim_{x\to 3} \frac{f'(3)(x-3)}{g'(3)(x-3)}$$