Children's Understanding of Numerical Variables

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CHILDREN'S UNDERSTANDING OF NUMERICAL VARIABLES

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The investigation of children's understanding of generalised arithmetic is one of about 10 studies of secondary school mathematics being undertaken by the maths wing of the CSMS project at Chelsea. The results reported here were obtained by giving a half-hour pencil and paper test (Algebra 1) to 3 000 secondary school children in the summer of 1976. Most of the children were in the second, third or fourth year of secondary school, so their mean ages would have been about 13.3, 14.3 and 15.3 respectively. (In what follows, third-year percentages will be quoted throughout.) The children came from 12 comprehensive and three selective schools and were chosen in such a way that their IQ distribution (for each year group) did not differ significantly from what one would expect for a representative sample of children in English schools.

The test itself was developed by first interviewing children individually (about 30) and then trying pencil and paper versions on a few classes at a time. In all, the test went through 10 drafts.

Variables, from the Child's Viewpoint

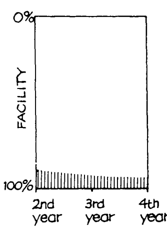
As mathematics teachers, we have the tendency to use the blanket term "variable" for any and all letters in generalised arithmetic. However, if g is a variable in "$f=3g+1$; what happens to f if g is increased by 2?", then in what way is it sensible to call g a variable here: "if $g+5=8$, $g=$. . ."? From the mathematicians point of view the term may (technically) apply in both cases, but consider the child: only 7% of third years answered the first question correctly, and 92% correctly answered the second. The two questions are so vastly different in cognitive demand that the term "variable" can only serve to obscure this difference.

Collis (1975*a*) identified several ways in which children interpret letters in generalised arithmetic, and in so doing he considerably sharpened and refined the meaning of "variable". His ideas formed an important basis for constructing the Algebra test, and have been developed further to give the following six levels for describing the different ways the letters can be used:

Letter EVALUATED
Letter IGNORED
Letter as OBJECT
Letter as SPECIFIC UNKNOWN
Letter as GENERALISED NUMBER
Letter as VARIABLE.

An example of an appropriate item for each level is shown below; a fuller description follows after the next section.

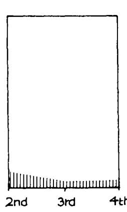
Letter EVALUATED



$a+5=8$
$a=?$

Here a can be evaluated immediately; there are no intermediate steps involving an unknown.

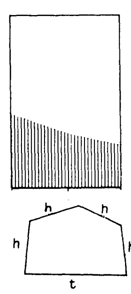
Letter IGNORED



$a+b=43$
$a+b+2=?$

The second equation differs from the first by the term $+2$; $a+b$ can be ignored.

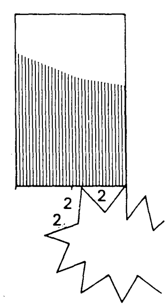
Letter as OBJECT



$p=$. . . . .

h and t are names or labels for the sides, rather than numbers.

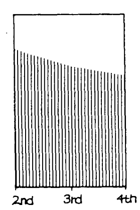
Letter as SPECIFIC UNKN



Part of this figure is not drawn. There are n *sides altogether, all of length 2.* $p=$. . . . . .

n stands for an unknown number which cannot be evaluated.

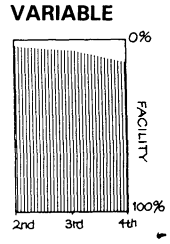
Letter as GNRLSD NMBR



$c+d=10$
$c<d$
$c=?$

c represents a set of numbers rather than just one value.

Letter as VARIABLE



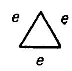
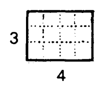
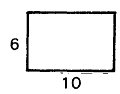
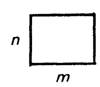
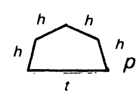
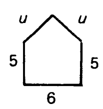
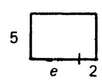
Which is larger, $2n$ or $n+2$?

A second order relationship needs to be found between $2n$ and $n+2$ as n varies.

Piagetian Stages

The test was constructed within a Piagetian framework, and an attempt has been made to classify items in terms of Piaget's stages of concrete and formal operational thought. Broadly speaking, where the last three levels (above) apply, items require formal operations; however, consideration must be given to the context in which the levels are being used: the child's likelihood of solving any given item will depend not only on the level of interpretation that the item requires but on the interaction of this with other dimensions such as the type of operation involved (Brown and Küchemann, 1976), the number of operations (Collis, 1975*b*), and so on — in other words, on the *complexity* of the item.

The table below lists a selection of the 51 items in the test, and shows the Piagetian sub-stage of each item, the level of interpretation at which the item can be solved successfully, and also some common wrong answers. A further discussion of the levels of interpretation follows, using these items as examples.

Piagetian Sub-Stage	Item	Question Number on test	% Correct	EVALUATED	IGNORED	OBJECT	SPECIFIC UNK	GNRLSD NMBR	VARIABLE	QUESTION	Common Wrong Answers	%
Early Concrete	A	5i	97		x					If $a+b=43$, $a+b+2=\ldots$		
	B	9i	94			x				 $p=\ldots$		
	C	6i	92	x						If $a+5=8$, $a=\ldots$		
	D	7i	91							 $A=\ldots$		
	E	7ii	89							 $A=\ldots$		
	F	1i	88							If $x\to x+2$, $6\to\ldots$		
Late Concrete	G	5ii	74		x					If $n-246=762$, $n-247=\ldots$	763	13
	H	2	72		x					smallest, largest of: $n+1$, $n+4$, $n-3$, n, $n-7$		
	I	4i	68		x					Add 4 onto $n+5$	9	20
	J	7iii	68			x				 $A=\ldots$		
	K	9ii	68			x				 $p=\ldots$	$p=4ht$ or $hhhht$	20
	L	9iii	64			x				 $p=\ldots$	$p=2u16$ or $uu556$	16
	M	11ii	62	x						If $m=3n+1$ and $n=4$, $m=\ldots$		
	N	11i	61	x						If $u=v+3$ and $v=1$, $u=\ldots$	2	14
Early Formal	O	5iii	41				x			If $e+f=8$, $e+f+g=\ldots$	12	26
	P	14	41				x			If $r=s+t$ and $r+s+t=30$, $r=\ldots$	10	21
	Q	9iv	38				x			n-sided polygon, each side of length 2; $p=\ldots$	36, 38, etc.	18
	R	4ii	36				x			Add 4 onto $3n$	$7n$	31
	S	16	30					x		What can you say about c if $c+d=10$ and c is less than d	4 only	39
	T	18ii	25					x		Is $L+M+N=L+P+N$ always, sometimes or never true?	never	51
	U	20	22				x			Cakes cost c pence each and buns cost b pence each. If I buy 4 cakes and 3 buns, what does $4c+3b$ stand for?	4 cakes and 3 buns	39
Late Formal	V	4iii	17				x			Multiply $n+5$ by 4.	$n+20$	31
	W	7iv	12				x			 $A=\ldots$	$e+10$, $10e$, $7e$	28
	X	22	11				x			"blue and red pencils" (see text).	$b+r=90$	17
	Y	17i	5				x			Mary's basic wage is £20 per week. She is also paid another £2 for each hour of overtime that she works. If h stands for the number of hours of overtime that she works, and if W stands for her *total* wage (in £'s) write down an equation connecting W and h:	$W+h$ or $W=20+h$	27
	Z	3	6						x	Which is larger, $2n$ or $n+2$? Explain.	$2n$	71

Letter EVALUATED

In item C(92%) the numerical value of a can be directly determined by simple trial and error; there is no step at which a has to be handled as an unknown. M(62%) and N(61%) are more complex but still only require concrete operations.

Letter IGNORED

Consider Question 5 (items A, G and O): Item A(97%) may look complicated (2 variables?) but in fact is answered correctly by nearly all the children. The expression $a+b$ occurs in the same way in both equations and, by means of a "matching" technique (Collis, 1975*a*) can essentially be ignored. The child needs only to focus on the operation $+2$, which is the only difference between the two equations, and apply the operation to 43. It is perhaps not quite precise to say $a+b$ has been ignored (but it's the best term I could come up with): the child perhaps has to note the existence of $a+b$, but having done so $a+b$ can be put to one side; there is no need to handle, transform or even remember the expression.

Question 5

If $a+b=43$	If $n-246=762$	If $e+f=8$
$a+b+2=\ldots\ldots$	$n-247=\ldots\ldots$	$e+f+g=\ldots\ldots$
Item A	Item G	Item O

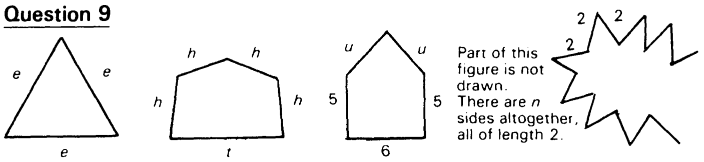
G(74%) is more difficult: the numbers are larger, the operation (-1) is implicit and involves subtraction rather than addition, but otherwise the item is of the same form as A and again the letter can be ignored. Item O(41%) is also of the same form but this time, though $e+f$ can be ignored, g cannot. g is an unknown number which cannot be evaluated but which has to be used (specific unknown). The answer $8+g$ is in a sense "incomplete", "unresolved" and Collis calls the willingness to cope with an expression of this type "Acceptance of Lack of Closure", which he regards as an important indicator of formal operational thought (see also Lunzer, 1976). Most of the children tested could not cope with $8+g$ and many tried to resolve the problem (produce closure) by finding plausible values for g; thus 26% gave the answer 12 $(4+4+4=12)$, while others gave 9 (add 1), 10 (?) and even 15 (g is the seventh letter of the alphabet, $8+7=15$).

Letter as OBJECT

At this level the letter can be operated upon without first having to be evaluated, but the letter is regarded not as an unknown number but as an object or a name or shorthand for an object. (This notion is quite often introduced to children when, say, the simplification of $2a+5a+3b$ to $7a+3b$ is explained in terms of "a stands for apples, b for bananas" — which Galvin and Bell, 1977, have aptly called "fruit salad algebra".)

Consider Question 9, (items B, K, L and Q). In B(94%) and K(68%) the letters can successfully be treated just as labels for the sides, which simply have to be listed or collected together. Thus many children (20%) gave answers like $p=4h$, t or $p=hhhht$ and even for item L(64%), which contains actual numbers as well as letters, children gave answers of the type $p=2u+2.5+1.6$ — very much an act of collecting.

Question 9



$p=\ldots\ldots\ldots$	$p=\ldots\ldots\ldots$	$p=\ldots\ldots\ldots$	$p=\ldots\ldots\ldots$
Item B	Item K	Item L	Item Q

Item Q(38%) is much more difficult and requires formal operations. Though the answer is of the same form as that of item B ($p=2n$ or $n2$, as against $p=3e$) the level of letter as object is no longer adequate: n is clearly a number (specific unknown). As with the answer $8+g$ discussed earlier, most children cannot cope with the lack of closure that $p=2n$ requires and many literally closed the figure by drawing in a few more sides, thus arriving at $p=36$ or $p=38$, etc (18%).

Letters were frequently (mis)used as objects when a problem involving quantities had to be translated into mathematical language, and when a mathematical statement had to be interpreted (see also Galvin and Bell, 1977). Thus consider item X(11%). The most common answer was $b+r=90$ (17%), which might be taken as an abbreviation of "*b*lue pencils plus *r*ed pencils cost 90p" rather than a relationship between the numbers b and r. Occasionally children found correct numerical values for b and r, for example $b=6$ and $r=10$, but then wrote $6b+10r=90$, i.e. "6 *b*lue pencils and 10 *r*ed pencils cost 90p".

Item X

Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence.

If b is the number of blue pencils bought, and
if r is the number of red pencils bought,
what can you write down about b or r? ....

Question 10 proved even more difficult. It *is* a mean question (mixed veg algebra?) but only 1% of children gave a clear correct answer to the first part, and only 4% to the second. Most children thought $8c+6t$ meant "8 cabbages and 6 turnips" (52%); 23% gave the answer £1. And for the total number of vegetables most wrote 14(72%) instead of $c+t$.

Question 10

Cabbages cost 8 pence each and turnips cost 6 pence each.

If c stands for the *number* of cabbages bought
and t stands for the *number* of turnips bought,
what does $8c+6t$ stand for? ....

What is the total number of vegetables bought? ....

Letter as SPECIFIC UNKNOWN

At this level the letter is thought of as a specific, albeit unknown, number which can be operated upon without having to be evaluated. Items O(41%) and Q(38%) which have already been discussed require this level, as do items R(36%) and the far more complex V(17%) of Question 4; however, I(68%) of Question 4 does not.

Question 4

4 added to n can be written as $n+4$. Add 4 onto each of these:			*n multiplied by 4* can be written as $4n$. Multiply each of these by 4:		
8	$n+5$	$3n$	8	$n+5$	$3n$
......	......	......	......	......	......
	Item I	Item R		Item V	

The dominant wrong answer for R is interesting: 31% of children gave the answer $7n$ instead of $3n+4$. This seems to be arrived at by a kind of "association", whereby the elements in the item are simply joined in the most immediately obvious way, without reference to what the elements might represent: 3 and 4 give 7, and the n is just tagged on the end. Thus essentially the letter is ignored, and in fact a further 16% simply gave the answer 7, so ignoring n entirely. Similarly for V, 31% gave the answer $n+20$ and 15% just wrote 20. Notice, however, that this strategy is quite adequate for item I, where 68% correctly gave the answer $n+9$.

All the items so far discussed under the heading of specific unknown have involved *answers* that require Acceptance of Lack of Closure ($8+g$, $2n$, $3n+4$, etc.). It can be argued that children have an understandable reluctance to provide answers of this "incomplete" sort, but that this does not demonstrate their inability to use specific unknowns *per se*. Thus it is of interest that item P(41%), which can be solved by substituting r for $s+t$ in the second equation, but where the answer is numerical ($r=15$), is of comparable difficulty.

Item P

What can you say about r if $r=s+t$ and $r+s+t=30$

Letter as GENERALISED NUMBER

This level differs from specific unknown in as much as the letter is seen as being able to take, or as representing, a series of values rather than one value only. In item S, 30% gave the answer $c<5$ or a systematic list like 1, 2, 3, 4, but the most common answer was just a single value for c, usually $c=4$ (39%).

Item S

What can you say about c if $c+d=10$ and c is less than d

Letter as VARIABLE

Interpreting letters as variables involves an awareness that there is some kind of relationship between the letters, as their value changes in a systematic manner. Consider again item X: the answer here is $5b+6r=90$ which can be arrived at, and interpreted, in a variety of ways not necessarily involving letters as variables. For example, b and r can be thought of as specific unknowns: b is *the* number of blue pencils bought (which I do not happen to know at the moment), and so the cost of blue pencils is given by "the number of pence that one blue pencil costs" × "*the* number of blue pencils bought" ($5\times b$), and similarly for the red pencils.

Item X

Blue pencils cost 5 pence each and red pencils cost 6 pence each.
I buy some blue and some red pencils and altogether it costs me 90 pence.

If b is the number of blue pencils bought, and
if r is the number of red pencils bought,
what can you write down about b and r? ....

Alternatively, one might start by listing pairs of values that satisfy the constraints of the question: b and r could be 6,10 or 12,5 or 0,15, etc., in which case $5b+6r=90$ can be seen as a generalisation of these values and b and r seen as generalised numbers. At this level each pair, *in turn*, satisfies $5b+6r=90$ but the pairs are not considered as a whole: for example they are not ordered into, say, 0,5 6,10 12,5. It is this "co-ordination", or "structuring" of the values, leading to the additional insight that, for example, "as b increases, r decreases" which occurs at the level of letter as variable.

The item that best tests this level of understanding is probably item Z(6%). 71% of children wrote that $2n$ was larger than $n+2$, usually for a reason such as "because it is multiply", which, intuitively, most of us would probably agree with. Other children chose a value for n, say $n=5$, and inferred from the single case of 10>7 that $2n>n+2$. (There were also nice answers like "$2n$ is bigger because $n+2$ is smaller".)

Item Z

Which is the larger, $2n$ or $n+2$? ....

Explain: ....

The key to the item is the fact that, when $n=2$, the two expressions are equal, but how is the child to discover this? Consider a child who can cope with generalised numbers and so chooses several values for n, say $n=5$ and $n=9$, which give 10,7 and 18,11 for $2n$ and $n+2$: each (isolated) case again supports the conclusion that $2n>n+2$ and most children would be satisfied that this was an adequate investigation of the relationship between $2n$ and $n+2$. However, there is also a relationship between the cases 10>7 and 18>11, namely that the difference between $2n$ and $n+2$ increases as n increases ($18-11>10-7$). Thus the relationship between $2n$ and $n+2$ is actually changing with n, and it is with this awareness, of a "second order relationship" (or second order operation — Inhelder and Piaget, 1958), that n can be thought of as a variable, and which opens up the possibility that, for some n, $2n$ may equal or even be less that $n+2$.

Discussion

There is always the danger that these and other CSMS findings will add fuel to the "Great falling standards Debate". However, the purpose of the Algebra test is not to measure performance of mathematical techniques and algorithms, but rather to get a better idea of the way children cope with certain mathematical problems.

It is hoped that the Algebra results will underline the urgent need to improve the match between children's understanding and the mathematical demands that we as teachers put on the children. More specifically, and on a day-to-day basis, the test might provide some kind of framework within which teachers can interpret their pupils' efforts.