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Journal of Mathematical Behavior 23 (2004) 371–388

Mathematical Behavior

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Supporting the development of algebraic thinking in middle school: a closer look at students' informal strategies

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Available online 3 October 2004

Abstract

This study investigated how 31 sixth-, seventh-, and eighth-grade middle school students who had not previously, nor were currently taking a formal Algebra course, approached word problems of an algebraic nature. Specifically, these algebraic word problems were of the form $x + (x + a) + (x + b) = c$ or $ax + bx + cx = d$. An examination of students' understanding of the relationships expressed in the problems and how they used this information to solve problems was conducted. Data included the students' written responses to problems, field notes of researcher–student interactions while working on the problems, and follow-up interviews. Results showed that students had many informal strategies for solving the problems with systematic guess and check being the most common approach. Analysis of researcher–student interactions while working on the problems revealed ways in which students struggled to engage in the problems. Support mechanisms for students who struggle with these problems are suggested. Finally, implications are provided for drawing upon students' informal and intuitive knowledge to support the development of algebraic thinking.

Keywords: Algebra; Algebraic thinking; Middle school; Guess and check

The rethinking of algebra has introduced new conceptions about what algebra and algebraic thinking might be and how it can be developed. This "rethinking" had led to a broader conception of algebra allowing for the inclusion of activities and perspectives that were not previously thought of as algebra (Kieran, 1996). For example, the *Principles and Standards* (National Council of Teachers of Mathematics [NCTM], 2000) suggest that algebra be thought of as a strand that is spread across the K-12 years rather

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0732-3123/$ – see front matter
doi:10.1016/j.jmathb.2004.09.001

than as a series of courses such as Algebra and Pre-Algebra. In turn, they support the development of programs and approaches to algebra that will help bring elementary and middle school students into algebra in such a way as to make formal algebra and the explicit and implicit thinking associated with it accessible to all.

Typically, school algebra has provided a limited perspective of algebra focusing on the "describe and calculate" (Lesh, Post, & Behr, 1987) activities of algebra. This limited view focuses on "doing" algebra through writing and manipulating equations. It does not include thinking about "using" algebra, or acknowledge activities that give meaning to traditional algebra activities. Kieran (1996) argues that there are three types of algebra-based activities. First, algebra involves "generational" activities where situations are generated into equations or expressions. Second, there are "transformational" or rule-based activities such as collecting like terms, factoring and simplifying expressions. Third, there are "global, meta-level activities" where algebra is used as a tool but this use does not have to be restricted to algebra.

The "global, meta-level" activities involve understanding the relationships among quantities that underlie a situation. It includes representations that are not necessarily letter-symbolic but allow one to handle quantitative situations in a relational way. This does not mean that students should not learn about the generational or transformational aspects of algebra, but that they might be approached differently. Kieran (1996) writes:

Thus, algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourses of school algebra. (p. 5)

This notion of algebraic thinking is useful to think about when creating experiences that provide rich contexts for discussion during the elementary and middle school years. Similar to Kieran, Driscoll (1999) argues that at times students' informal thinking is not that different from formal symbolic approaches. He suggests that prior to algebra, teachers should look for opportunities to develop in students the "habits of mind" that people use when they think algebraically.

Research indicates that students with and without algebra backgrounds have informal strategies that allow them to effectively deal with algebraic word problems or problems that are start-unknown (Bednarz & Janvier, 1996; Hall, Kibler, Wenger, & Truxaw, 1989; Kieran, Boileau, & Garancon, 1996; Nathan & Koedinger, 2000; Rojano, 1996). Two of the most noted approaches are guess and check and the unwind strategy. The following problem provides a context to discuss these two approaches: A middle school raised \$3000 for a field trip. The sixth graders raised \$400 more than the seventh graders and the eighth graders raised \$200 more than the seventh graders. How much did each grade raise?

Students who guess and check choose a value to represent an unknown quantity and they test or check the accuracy of the guessed value using relational reasoning. They apply the mathematical operations described or implied in the context of the word problem to see if their guess leads to the targeted total. If not, a new guess is made and the trial is repeated by applying the same relational reasoning, checking to see if the new guess will produce the desired total. In the fundraiser problem a student might guess an amount for the seventh-grade class, add \$400 to this to find the amount the sixth-grade class raised, and add \$200 to the original guess for the seventh-grade class to find the amount the eighth-grade class raised. In order to check if these are the right amounts for each grade, the student then adds the amount each class raised to see if it totals \$3000. If not, a new guess is made and the same relational reasoning is applied to see if it leads to \$3000. Ideally, students will use the information gained from previous guesses to make a reasonable new guess that will systematically move them toward the correct solution.

A second approach is the unwind method. Students using this approach take the structure of the problem into account and use inverse operations to work backwards and undo or "unwind" the mathematical constraints and isolate the unknown quantity. For example, with the fundraising problem, one would subtract from $3000 each of the known quantities to undo the addition of each the $400 and the $200. This leaves $2400 and three unknown but equal quantities that the description for each grade's effort is based on. If the $2400 is divided by 3, it is revealed that the seventh-graders raised $800, the sixth graders raised $400 more than this with $1200, and the eight-graders raised $1000 which is $200 more than the seventh-grader's $800.

While these approaches are often labeled as arithmetic, they can also be viewed as ways in which students make sense of algebraic situations without having to rely upon the traditional forms that are found in school algebra. Rojano (1996) points out that strategies such as trial and error or guess and check are sometimes seen as obstacles to learning algebraic methods, but argues that "trial and error, together with other strategies considered informal and found in students beginning the study of algebra, are indeed a real foundation upon which the methods or strategies of algebraic thought are constructed" (p. 137).

Similarly, Driscoll (1999) suggests that the boundaries in the meaning of algebraic thinking need to be more permeable. He argues that "facility with algebraic thinking includes being able to think about functions and how they work, and to think about the impact that a system's structure has on calculations" (p. 1). Speaking specifically about guess and check, he explains that students can guess a number, and test to see what type of output it gives. By reevaluating and trying other numbers, students are numerically testing out a more general algorithmic statement that they think is representative of the situation in the problem. This type of reasoning is representative of the reasoning one uses when "building rules to represent functions".

Hall et al. (1989) found that when college undergraduates who had successfully completed their high school algebra course were presented with non-routine algebra problems they often used model-based reasoning where they worked with the situational context presented in the problem to solve algebra problems and did not draw upon the "algebraic" techniques or the generational and transformational activities they learned in algebra. Hall and colleagues argue that activities such as guess and check, a form of model-based reasoning, are legitimate in their own right and instruction that foregrounds the formalisms of algebra and disregards such reasoning needs to be reconsidered. Algebraic formalisms are of little use if the problem solver does not understand the problem's structure.

The first goal of this research is to investigate how sixth-, seventh-, and eighth-grade students, who have not been exposed to formal algebraic methods for solving the proposed word problems, approach and solve these problems. What types of reasoning do students use to solve these problems? A second goal of this research is to explore what types of instructional support students might need to make sense of these kinds of problems. What types of questions do they ask when they work on these problems? What types of interventions were used and were they successful? A final goal is to consider the students' reasoning in terms of the broadening perspective of algebra and algebraic thinking previously offered, by discussing ways in which students' informal reasoning and approaches can provide cognitive support for developing the thinking patterns and more traditional discourse of algebra.

1. Design of study

1.1. Participants and setting

The participants were 31 middle school students from 3 middle schools in a large urban area. The 11 sixth graders attended one school, the 13 seventh graders another school, and the 7 eighth graders yet another. The students were described by their teachers as average students and were not placed in an accelerated or above average class. The students did not have prior instruction on algebraic methods for solving the word problems used in the study.

For each grade, one teacher provided access to students from two different classes. Each class worked on a different algebra word problem. While class size averaged from 20 to 25 students, a low number returned the permission slips. Since the numbers varied in each class, the balance between boys and girls, as well as the number of students at each grade level, and the number of students who worked on each problem is uneven. Table 1 provides a breakdown of the participants for each problem. Having each student work both algebra problems was considered but led to a problem with scheduling: it would take more time, possibly more than one class period, for students to solve and write a solution to two problems. Some teachers were uncomfortable with taking students away from the planned curriculum material that the district expected them to cover for more than one class period. In addition, I wanted students to work and write without feeling pressured to complete both problems in the length of a class period.

1.2. Data collection

Data was collected from three sources: (1) field notes taken while students were working on the problems, (2) written work from each of the 31 students and (3) interviews with 18 students: 6 sixth graders, 7 seventh graders, and 5 eighth graders. Students were presented one of the following two problems.

> *Candy Bar Problem*: There were three kinds of candy bars being sold at the concession stand during the Friday dance. There were 22 more Snickers bars sold than Kit Kat bars and there were 32 more Reese's Peanut Butter Cups sold than Kit Kat bars. There were 306 candy bars sold in all. How many of each kind of candy bar was sold?
>
> *The Pet Problem*: There are 280 animals signed up to be in the pet show. There are four times as many dogs as cats and there are twice as many birds as cats. How many of each animal signed up for the pet show?

Table 1
Breakdown of participants by grade and gender for each problem

Grade level	Pet Problem		Candy Bar Problem	
	Male	Female	Male	Female
Sixth	2	6	3	0
Seventh	3	5	1	4
Eighth	1	2	2	2

The Candy Bar Problem involves additive relationships while the Pet Problem uses multiplicative relationships. These problems were chosen for two reasons. First, both are start-unknown and while they can be solved using direct-calculation approaches of arithmetic, they can also be solved using standard algebraic methods making them algebra level problems. Second, when expressed algebraically, the underlying structure has the form $ax + bx + cx = d$ and $x + (x + a) + (x + b) = c$. Filloy and Rojano (1984) have reported that problems of the form $x + a = b$, $ax = b$, or $ax + b = c$ can be easily solved by students using arithmetic methods such as inverse operations. By choosing a form with a more complex linkage of relationships, it was hoped that students would have to shift from using inverse operations to thinking about forward operations. At the same time, these problems were perceived by this researcher to be achievable for the students participating in this study.

The word problem and work expectations were presented to each class of students as a whole. A copy of the problem was given to the students and the researcher read the problem. Using the Pet Problem as an example, the problem was read twice followed by a statement about what information the problem contained.

From reading the problem we know that there are twice as many birds as cats and there are four times as many dogs as cats. We also know there are 280 animals altogether. Knowing this information, you are asked to find out how many of each animal signed up for the pet show.

Students were asked to not erase what they tried that did not work. When they found a solution they were asked to explain their reasoning in writing. The following scenario was presented to help them explain their reasoning, “If you had to explain to a friend what you did and what you were thinking when you worked on the problem, what would you say to help them understand what you did?”

Once introduced, students were given approximately 25 min to work on the one problem. Students were given calculators to use. While participants worked on the problem, the researcher observed, took notes and answered students’ questions individually. When a student asked the researcher a question, an intervention occurred. Asking questions was not discouraged so that data could be collected on the types of question students might ask and what effect an intervention might have.

Finally, selected students were interviewed later that day or the next day. The interviews were audio-taped. Of the 31 students who participated in the study, 19 students worked on the Pet Problem and 8 were interviewed. There were 12 students who worked on the Candy Bar Problem, of which 12 were interviewed. Two ideas guided the choice of whom to interview. First, some students were able to write very clear narratives describing what they were doing. These students were not interviewed. Rather, students with no narrative or unclear narratives were chosen. Second, students with interesting approaches were chosen. For example, one student who was not successful in solving the problem made a grid and was trying to use ratios to solve the problem.

1.3. Data analysis

The first analysis of the data focused on ways to characterize the students’ different responses to the word problems. Initially the responses were categorized as systematic guess and check, random guess and check, unwind and other. It was also noted whether students were successful or not. Due to the large number of students using systematic guess and check, both successful and not, the data was re-categorized and a second analysis was done. In the second analysis the responses were categorized as successful systematic guess and check, unsuccessful systematic guess and check, successful alternative approaches, and unsuccessful alternative approaches. The "alternative" category includes approaches such as unwinding, random number manipulation, and students who appeared to use guess and check but it was not clear if it was systematic.

In order to make claims about how students using the same approach reasoned about a problem, a characterization of each student's solution approach was created. Each student's work was analyzed and evidence for the claimed reasoning was recorded by corresponding the claim to the place in the student work, interview transcript or field notes where the claim was supported. As part of the analysis, five specific questions were addressed: (1) Were students aware of the relationship between the objects in the problem? (2) Did students have a clear approach or was their work random? (3) Did students have a specific item (for example, Snickers bars or dogs) that they started with when they were solving? (4) Did students randomly play with the numbers given in the problem before they settled in on an approach? and (5) Did the researcher intervene when a student was working on a problem? If an intervention occurred the nature of the intervention was recorded with the characterization of their approach to the problem so that it could be determined what effect an intervention might have had on a student response.

2. Findings: understanding student reasoning

When looking at the breakdown of approaches, Table 2 shows that close to two-thirds of the students were successful in solving the problems with systematic guess and check emerging as the most common approach. Of the 20 students that were successful, 16 used a systematic guess and check approach and 4 used alternative approaches. Of the 11 that were unsuccessful, 3 used a systematic guess and check approach and 8 used alternative approaches.

In this section the reasoning that emerged with each of the four different approach categories will be presented and discussed. Since being successful, or not, was built into the categorization, what students understood about the problems as well as the difficulties students faced will be examined. For each approach, any interventions that occurred will also be discussed.

Table 2
Frequency of approaches and success rate

	Successful		Unsuccessful	
	Systematic guess and check	Alternative	Systematic guess and check	Alternative
Sixth grade				
Pet Problem	1	0	2	5
Candy Bar Problem	3	0	0	0
Seventh grade				
Pet Problem	6	1	0	1
Candy Bar Problem	1	2	1	1
Eighth grade				
Pet Problem	2	1	0	0
Candy Bar Problem	3	0	0	1
Total	16	4	3	8

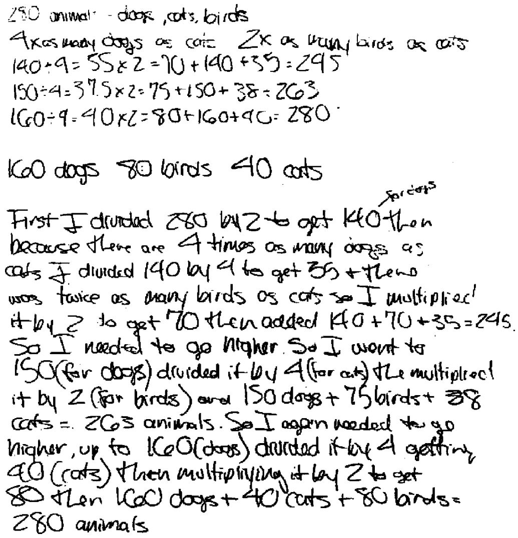

Fig. 1. Divide by 4, times by 2 system used by seventh-grade student.

2.1. *Successful approaches*

2.1.1. *Successful systematic guess and check*

Of the 31 participants, 16 successfully used systematic guess and check. To be classified as systematic guess and check there had to be clear evidence that the student had a specific algorithmic system they used when guessing and checking and that they were systematically adjusting the amount of the guess based on what they found when they checked. There were three different guess and check systems used by students working on the Pet Problem and three used with the Candy Bar Problem.

Three students used the "divide by 4, times by 2" system to reason about the Pet Problem. The seventh grader whose work is displayed in Fig. 1, shows that she has a clear algorithmic system and that she is systematically using it. First, she chose a number for dogs, divided by 4 to get the number of cats, then multiplied the number of cats by 2 to get the number of birds. Finally she added the three amounts to see if the total was 280. When the total was not 280, she used the total to adjust her guess. After getting a total of 245 she writes, "So I needed to go higher, so I went to 150 (for dogs) . . .". Another seventh-grade student who used the "divide by 4, times by 2" approach drew upon knowledge of factors and multiples when choosing the initial value. He writes, "I kept finding multiples of four for the number of dogs because you have to divide it by 4 to get the number of cats".

Five students used a "times 4, times 2" system. The sixth grader whose work is shown in Fig. 2 started by guessing a number for cats and multiplying by 4 to get the number of dogs. Next, she multiplied the number of cats by 2 to find the number of birds. Finally, she added the numbers to see if the total was 280. She wrote, "and when I added my numbers together I could see how much more I needed so I either added some cats or subtracted some . . ." She, like the other successful guess and check students, systematically used the results to adjust the starting values that were fed into the algorithmic system.

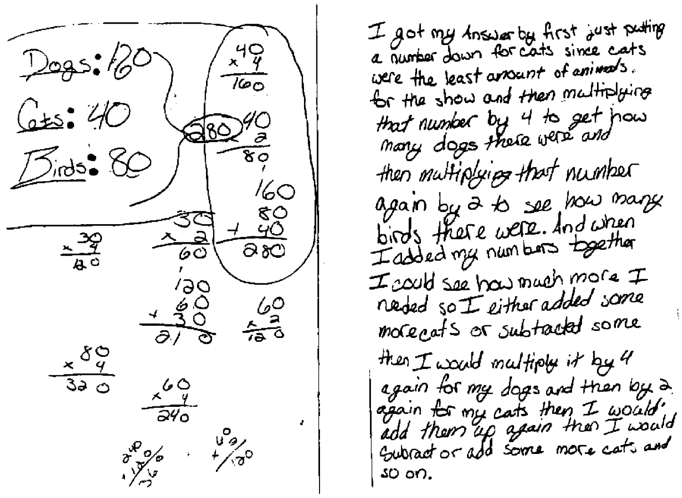

Fig. 2. Times 4, times 2 system used by sixth-grade student.

One eighth-grade student used a “times 2, times 2” system. As she talks about her system during an interview, she shows that she has a clear, correct, but unexpected view of how the unknown values (number of pets) are related. This student uses the number of cats as the starting point. She explains that she used cats because they are “the lowest number of animals in the pet show”. With her system she chooses a number for cats, doubles it to find the number of birds “because there are twice as many birds as cats”. Next, she takes the number of cats and doubles it. Rather than working with the explicitly described relationship of 4 times as many dogs as cats, she explains that “after I times the cats by 2 to get the birds and the dogs is 4 times as many cats, then the dogs must be two times as many as birds”.

The three different systems used to solve the Candy Bar Problem also illustrated that students were able to link the relationships between unknown number of candy bars and the total to systematically find the correct number of each candy bar. Five of the seven students used an “add 22, add 32” system. They took the number of Kit Kat bars and added 22 to find the number of Snickers bars. Next, they took the number of Kit Kat bars and added 32 to find the number of Reese’s Peanut Butter Cups. Finally, the amounts for the three candy bars were totaled to see if the sum was 306. One seventh-grader developed a different system where he began by choosing a number for the Reese’s and subtracted 32 to find the number of Kit Kats. He then added 22 to the Kit Kat number to find the number of Snickers bars. Finally, he totaled the three numbers to see if the total was 306. In a third approach, a sixth-grade student whose work is displayed in Fig. 3 realized that when using the “add 22, add 32” system[1] that he added the Kit Kat number three times when finding the total. He adjusted his system and chose a Kit Kat number, multiplied it by 3 and then added 22 and 32. Using 84 as the guess for the number of Kit Kats he explained when interviewed, “It would have been easier than having to add 22 and 32 [to the 84] and then add 84 . . .”.

[1] This student originally used the “add 22, add 32” system with 86 and 85 before settling on 83. He explains this in the interview. The original copy of his work indicates that he tried other numbers and was systematic. He erased his initial work, and while the erasures are visible on the original, they are not visible when the original is reproduced.

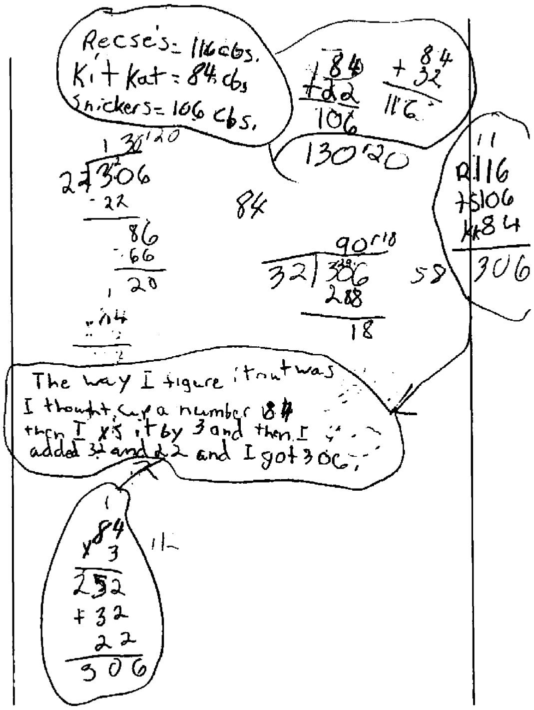

Fig. 3. Times 3, plus 22, plus 32 system used by sixth-grade student.

In all cases, these students were able to make a guess, apply their strategy and then systematically adjust their guess up or down according to the total they got.

2.1.2. Other alternative successful approaches

Four students used one of two alternative approaches successfully, unwind or random guess and check. Fig. 4 shows the work of the one student who used the unwind approach. While this student can clearly explain what calculations he made and describes what is accomplished by adding 22 and 32 to 84, he does not explain why it makes sense to initially add 32 and 22 and then subtract that sum from 306. When asked about this in an interview, he was not able to provide a reason. This is not meant to imply that he is characteristic of all students using an unwind approach, but he does not provide evidence that he has a developed understanding of why this approach works. His last step involved finding the quantity of each candy bar and checking to see if the three values totaled 306. The check, as used by this student, served as proof that the newfound numbers fit the relationships described in the problem rather than as a means for generating numbers that would total 306.

The other successful students used a random guess and check approach. A seventh-grade student who worked on the Candy Bar wrote the following on her paper, “Then I kept lowering the numbers to get 3 numbers that would equal to 306”. When interviewed this student was never able to clearly explain how she got the three numbers for the candy bars. The following interview dialogue indicated that her method was random:

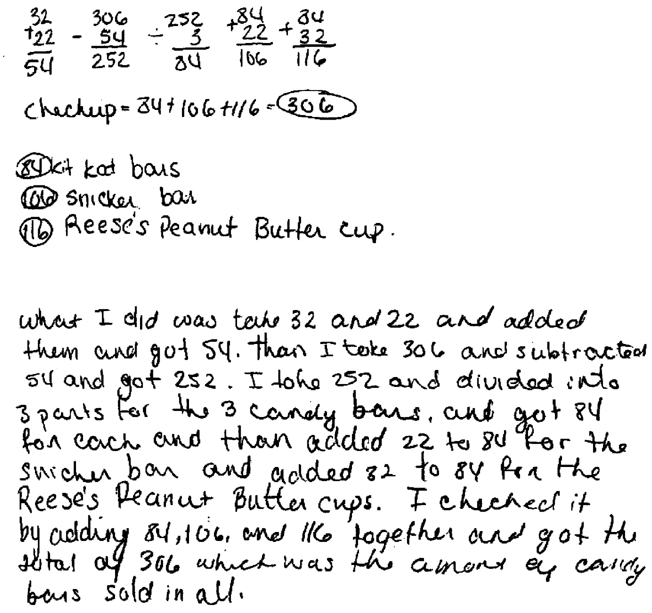

Fig. 4. Unwind approach used by seventh-grade student.

Student: Then I got the Kit Kat one and took 106 for the total and then I took away 84, because in between 116, 106 and 84 yeah — because 84 Kit Kats I took away and I got the difference of 32 between 84 and 116 and I did the same thing for 84 and 106 and got a difference of 22.

DJ: So you knew it fit the rules?

Student: Yeah.

DJ: Did you start by guessing numbers for all the candy bars and seeing if they fit the rules?

Student: Yeah. I would see if they had 22 more than Kit Kats and 32 more and it added up to 306.

First she focused on finding values for the unknown quantities that totaled 306 and then she checked to see if these numbers had the described relationships. This approach is also apparent in the narrative that an eighth-grade student wrote about his approach to the Pet Problem:

I used 120, 30, and 80 first but I add up wrong. Then I add ten to the cats. Then I see that the birds were two times (cats) the amount and I got my answer. Dogs 160, cats 40, and birds 80.

Like students using systematic guess and check, students using alternative approaches understood how the unknown quantities were related and took the total into account. But these students did not use the relationship among quantities to drive their work. These students first found quantities that totaled 306, and then checked to see if the relationships held. This is slightly different from the systematic guess and check students who first developed a system that they knew would generate values for the unknowns using the described relationships, and then checked to see if the total was correct.

2.1.3. Interventions with successful students

Of the 21 successful students, interventions occurred with one sixth-grade student and three eighth-grade students. Each of the four students were using systematic guess and check. In all interventions, students initiated the conversation about their work. My initial response was to read the problem to them and ask them if they had met the problem requirements and to have them talk about why or why not. Only when this did not help them did I intervene further.

A sixth-grade student became confused and incorrectly applied his "add 22, add 32" system. He chose 70 for the Kit Kats, added 32 to get 102 for the Reese's Peanut Butter Cups. Then he added 22 to the amount for Reese's Peanut Butter Cups, rather than Kit Kats, to get 134 for the Snickers. Since this totaled 306 he raised his hand. When I came over he said that he thought he might have figured out the answer. I responded by saying, "Let's see if the amounts agree with what the problem says". We examined his numbers as we reread through the problem description. He identified his mistake, which in the interview he described by explaining that he was supposed to add the 22 to the number of Kit Kats. After the interventions he correctly applied his "add 22, add 32" system to find the solution.

Some students initially tried to find three numbers that totaled 306 without attending to the relationship between the candy bars. When an eighth grader was working on the Candy Bar Problem, he arrived at 42 for Kit Kats, 82 for Snickers, and 182 for Reese's which totals 306. He called me over to look at his work. I reread the problem and asked if his quantities agreed with what the problem said. Not getting a response, I asked if his number for Snickers (82) was 22 more than the Kit Kat number (42). He said no. Not getting any additional response I offered, "If we use your 42 for Kit Kats, and think about what the problem says, what will the number of Snickers be?" He then added 42 and 22 to get 64. I reread the part of the problem that described the Kit Kat and Reese's relationship and asked if the number of Reese's was 32 more than the number of Kit Kats. Again he said no and then calculated that the number of Reese's would be 74. Without prompting he added the three numbers and got 180. I then had to prompt by asking, "What does that tell you about the number of Kit Kats?" He replied that he needed to try a higher number, which he then did. His written work showed that he then systematically applied the "add 22 add 32" approach to find the correct answer.

A third intervention occurred when an eighth-grade student working on the Pet Problem interpreted the multiplicative relationships as additive. Rather than find a number for birds that was twice the number of cats, she added two to the number of cats (63) to get the number of birds (65). She added four to the cats to get 67 for the dogs. Using a different context, candy bars instead of pets, we talked about what "twice" and "four times as many" meant. Initially she had a difficult time interpreting these relationships when I used large numbers. When I used 10 and asked what it would mean if she had 10 candy bars and I had twice as many, she was then able to figure out that twice (and four times) as many meant to multiply by 2 (and 4).

2.1.4. Characteristics of successfulness

Successful students were able to come to terms with both the relationships among quantities and the total. They were able to check their computation to determine if their answer met the problem specifications. For the students where interventions occurred, it is reasonable to assume that they would not have been able to move toward a solution without bringing the use of the relationships among quantities and the total to the forefront.

Students who used systematic guess and check interpreted the relationships between objects to create an algorithmic means for generating the total. They began by using relationships between unknowns and ended by checking to see if they had the correct total. In contrast, students using alternative approaches often found numbers that led to the correct total and then checked to see if the relationships held.

Student work and interviews indicated that some students tried random ideas prior to “settling into” an approach. For example, the student whose work appears in Fig. 3 first tried to divide 306 by 22 and by 32. Another student said she divided the total by 2 because she wanted to see what was happening. Other students divided the total by two or three to get a number to use as a reasonable first guess and then they began to investigate how to use the relationships. All in all, given adequate time to explore the problem context, and in some cases, with additional intervention, these students were able to make sense of and use the relationships among quantities and the total to solve the problem.

2.2. Unsuccessful approaches

2.2.1. Systematic guess and check

Of the 32 students, 3 students were not successful when using the systematic guess and check strategy. One seventh-grade student initially thought the three quantities in the Candy Bar Problem were equal. This was a case where an intervention occurred. About a third of the way through the work time the student raised her hand. When I came over she asked if the three “things” were equal. In response I read the problem statement to her and asked, “If the problem says that the number of Snickers is 22 more than the number of Kit Kats, would the amounts be the same?” The student reconsidered the problem and began to using the “plus 22, plus 32” system. She ran out of time before narrowing down the answer. During the interview she was able to explain her “plus 22, plus 32” system indicating that she was aware of how the unknown quantities and the total were related.

A sixth grade student working on the Pet Problem did not use a calculator and made several arithmetic errors. During the interview the following exchange occurred, illustrating that he began with the number of dogs and systematically employed a “divide by 4, divide by 2” system.

DJ: Did you pick a certain number first?

Student: The dogs.

DJ: How did you get the other numbers?

Student: I moved the 160 [his guess] down enough to get it two times and four times.

DJ: Okay, when you moved the 160 down what did you get?

Student: I got 90 but I do not think that is right.

DJ: And this [pointing to the 55 where he had 160, 90 and 55 written on his paper] is when you tried to go down four times?

Student: Yeah.

DJ: Were you using the calculator for this part?

Student: No.

DJ: If you had the calculator and you tried to make it go down what would you do?

Student: I would divide by two.

DJ: And if you wanted to get the other number [the number of cats] what would you do?

Student: Divide it [the number of dogs] by 4.

Despite having a working system in place, this student's effort was hampered by computational errors.

Another sixth-grade student working on the Pet Problem misinterpreted the multiplicative relationships as additive. Her paper contained 10 rounds of systematically choosing a number for dogs, subtracting 4 from the dogs to get the number of cats and then subtracting 2 from the dogs to get the number of birds. The number of dogs was adjusted according to whether the total was above or below 280. This system does not yield a solution that totals 280, so when she got as close as she could with a total of 279 (95 dogs, 91 cats, 93 birds), she disregarded the relationships and her "add 2, add 4" strategy and added one to the number of dogs so the total would be 280. Her final answer was 96 dogs, 91 cats and 93 birds.

2.2.2. Other alternative unsuccessful approaches

The eight students in this category used a variety of unsuccessful approaches. Several began with one approach and then shifted to another. Sometimes they showed an understanding of the relationships in the problems, and at other times they did not. There was a tendency to adjust the amount of the candy bars or animals by only attending to some, rather than all, of the relationships in the problem. There were also times when students forgot about the relationships and only focused on the total.

There were three students who began trying to use the relationship among the unknown quantities, but at some point disregarded the relationships and focused on the total. For example, an eighth-grade student tried two approaches to the Pet Problem. She began by using ratios and a 20 by 14 grid with 280 squares. She wrote ratios that correctly expressed the relationships between the cat and birds (1:2) and cats and dogs (1:4) on her paper. During an interview she explained that she began by dividing the grid into three equal parts of 93 (280 divided by 3) and then shifting squares around so the relationships were accounted for. "Since I figured that the cat was the smallest number because the dog and the bird were worth more . . . so I started taking away from the cat number [the squares in the grid for cat] and adding to the dog and the bird". When this did not work she switched strategies and chose a number for cats, 35 and multiplied it by two to get 70. She incorrectly labeled the 70 as dogs instead of birds. To find the number of birds she added the 35 and the 70 and then subtracted from 280 to see what was left to be the birds. Her written work shows that she was aware of the relationships among quantities. As she tried different approaches she switched from focusing on relationships to focusing on the total. She never checked her final attempt to see if the relationships held and was not able to successfully attend to both the relationships between unknowns and the total.

Four students were not able to demonstrate that they understood the relationships in the problem at all. Their work also indicated that they had underdeveloped understanding of operations. For example, one sixth-grade student who worked on the Pet Problem only used the "4 times as many" relationship. Her work in Fig. 5 illustrates that she has a weak understanding of multiplication concepts. For example she labels the action "times 4" as dogs. In another attempt, she multiplies 280 by 4, indicating that she does not understand that 280 is the total of all the animals.

2.2.3. Interventions with unsuccessful students

Of the 11 unsuccessful students, intervention occurred with one student who used systematic guess and check and with one student who used an alternative approach. The student who used systematic guess and check initially thought that the three candy bar quantities were equal was described in the last section. When she asked if the quantities were equal, I reread the problem and asked, "if one candy bar is 32 more and another is 22 more, would each of the candy bars be equal?" While the work shows that the

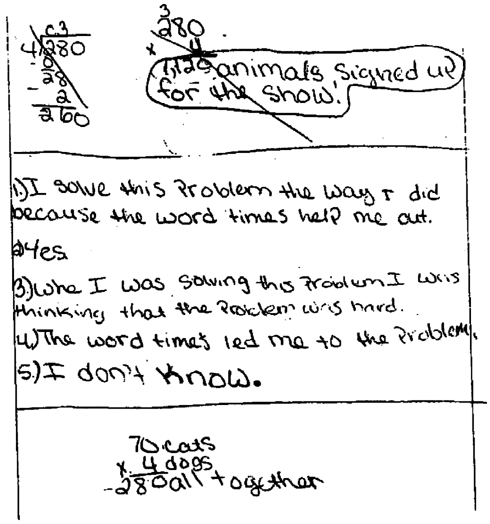

Fig. 5. Sixth-grade student who struggled to understand relationships.

student changed to using the systematic guess and check “plus 22, plus 32” approach, she was not able to produce the correct answer before the papers were collected.

An intervention with a student who used an alternative approach was not successful. This student struggled to understand what the problem was asking and the relationship among the quantities described in the candy bar problem. Rereading the problem to the student did not appear to be helpful. With prompting questions such as “What might the problem mean when it says there are 22 more Snickers than Kit Kats?” he was able to choose a number to add 22 and 32 to and find other quantities. Yet it is doubtful that he realized what the new quantities represented because he then only totaled the two quantities he had found and did not add in the quantity that was used to generate the other two. Looking at this work, we talked about how there were three types of candy bars. The students then used the numbers, 47, 15, and 37, Reese’s, Kit Kats, and Snickers, respectively. These three numbers correctly express the relationship between candy bars, but only total 99. When the student tried to adjust the quantities he only increased the number of Kit Kats, leaving the Reese’s quantity unchanged at 47 and the Snickers quantity unchanged at 37. Eventually, he arrived at a combination that totaled 306 but it did not take the relationship between candy bar quantities into account.

2.2.4. Characteristics of unsuccessfulness

Unsuccessful systematic guess and check students were hampered by computational errors, time constraints and confusion about the difference between additive and multiplicative relationships. Nonetheless, like their successful systematic guess and check counterparts, they were using an algorithmic approach that combined an understanding of the relationships among quantities and the total.

For unsuccessful students who used alternative approaches, there were a variety of reasons why they struggled. First, some students drew heavily upon arithmetic knowledge and conventions by only manipulating the given numbers. Their work did not indicate that they understood the relationships between quantities. These students randomly tried operations using the numbers in the problem. Second and more predominant, many unsuccessful students started by focusing on the relationships, but ultimately their focus moved to finding the correct total. While it may be that students experience difficulty keeping all aspects of the problem in the forefront, it also seems plausible that students' ability to think relationally is confounded by the use of arithmetic thinking where the focus is on finding the sum, difference, product or quotient. This type of reasoning was nicely captured by a sixth-grade student who upon reading the problem presented to her said, "I don't know how to do this because you don't have any numbers".

3. Discussion

The broad purpose of this study was to investigate how sixth-, seventh-, and eighth-grade students, those who have not been exposed to formal algebraic methods for solving the proposed word problems, approached and solved these problems. The results did not reveal any apparent grade-level differences or trends in approaches used. Several characterizations for students' approaches were revealed. First, most students were able to use both the relationships between the three unknowns and the total. Those who used systematic guess and check used the relationship among quantities to generate a value and then checked to see whether they had the correct total or not. Other students, such as those using the unwind approach or random guess and check, focused on using the total first and then checked to see if the relationships were correct. A third group of students began by using the relationship between quantities and the total, but eventually disregarded the relationships and focused on finding three values with the correct total. Finally, some students struggled to interpret the structure of the problem and never were able to utilize the relationship between unknowns at all.

While students had various methods for approaching these problems, systematic guess and check was the predominant method. Students who used systematic guess and check were clearly able to develop and articulate an algorithmic system where they took the relationships between the unknown quantities and the total into account. The systematic guess and check strategy is usually labeled as an arithmetic approach because students are manipulating numbers. Nonetheless, the thinking patterns used to design a guess and check system show students handling quantitative situations in a relational way which is characteristic of algebraic thinking as described in Kieran's (1996) broadened conception that includes global meta-level activities.

It is not so much that students guess, it is the thinking behind the guess that matters. In order to devise a system to systematically guess and check with, students had to understand the underlying structure of the problem and articulate it into a formal plan. Like Driscoll (1999) suggests, these students' written and interview responses indicate that they were building rules to represent functions. The students organized their data into a well-defined system in order to input values until the correct output was obtained. This type of rule-driven thinking, that which led students to design and use systems such as the "divide by 4, multiply by 2" system, is also used when one writes an algebra equation. Without any background in writing and solving algebraic equations, using their guess and check system as a function or input–output machine was a natural solution process for these students.

The results of this study differ from Bednarz and Janvier (1996) where seventh-graders solved similar problems using a greater variety of procedures and guess and check was used relatively little. However, there were several significant differences in the two studies including the magnitude of the study: one with a large number of subjects (132) versus a small number (31). In addition, Bednarz and Janvier looked at seven different types of linkages between quantities and this study only considered two. Bednarz and Janvier indicate that problem solving was not new to their students and I would offer that this is the case here as well. Perhaps the use of these two specific problem types, $x + (x + a) + (x + b) = c$ or $ax + bx + cx = d$, constrains the number of approaches and at the same time leads students to engage in the use of guess and check. Time may also be a factor. Students in this study were given 25 min to work on one problem with many using a majority of this time to solve the problem and write a solution.

The questions students asked and the interventions by the researcher may also provide insight into the results. While some of the students in this study did struggle to conceptually understand the problems, given time to engage in the problems, most of the students were able to decipher the situational context and use the relational information to solve the problem. The interventions revealed many ways in which students struggled to engage in the problems and utilize the relationships. For example, the data revealed that some students simply needed prompting in order to realize that the relationship among the unknown quantities needed to be taken into account. By rereading the problem and making the relationships explicit, students were able to focus and make sense of the problems. There were several students who made computational errors, but otherwise appeared to be moving productively toward solving their problem. For a few students, weak conceptual understanding of number and operations hindered their ability to engage sensibly in the problem. For these students it might be helpful to use problems with smaller quantities and two rather than three unknowns before moving to problems like those used in this study.

4. Conclusions and educational implications

Similar to Hall et al.'s (1989) work with undergraduates, the successful problem solvers in this study relied on various forms of model-based reasoning to engage in and solve algebra word problems. Understanding the structure of the problem was a key component in making sense of the problem thus indicating that students come to algebra with the potential for using global, meta-level reasoning. Based on these results, one might ask how instruction can legitimize and bring out the power in students' attendance to the situational relationships in these types of problems. In addition, while it is important to acknowledge and develop this type of reasoning, the power of algebra as a generational and transformational activity is also important and useful. Kieran (1996) suggests that it is important to combine literal symbolic and global, meta-level activities. This section draws upon the presented data to make a case for how instruction can address these two ideas.

Driscoll (1999) suggests that teachers can find algebraic potential in unexpected places. The responses these students developed provide opportunities to attach formal algebraic symbolism to students' natural language responses. For example, with the Candy Bar Problem, most students used the "plus 32, plus 22" system. But recall the sixth grade student whose work appeared in Fig. 3. In his paper he wrote that he "thought up a number 84 then I timesed it by three and then I added 32 and 22 and got 306". With either approach there is a potential opportunity to attach a formal representation to the student's informal representation in such a way that the symbols are used to represent mathematical ideas (NCTM, 2000). For example, the algorithmic system, "plus 32, plus 22" provides the basis for the symbolic representation $N + (N + 22) + (N + 32)$. The student who used the system "three times a number plus 22 plus 32", when stated in natural language, provides the basis for the symbolic representation $3N + 22 + 32$. Together, both representations provide a rich context in which to discuss how both are related to the context of the problem and to each other. There is also the opportunity to informally examine equivalent expressions and simplification of expressions.

In each of the systems described above, students used the same unknown items, Kit Kats, to generate their systems and guesses. Some students used other candy bars to represent the unknown *N*. Conversations about how the relationships can be interpreted in various ways, in terms of different unknowns, help students to understand that situations can be examined from various perspectives. Here the notion of using letters as a placeholder rather than as a variable comes to the forefront.

The student who used the unwind system to isolate the unknown quantity was actually employing the steps used to solve the equation $(3N) + 22 + 32 = 306$. In his solution, he first combined the like terms 32 and 22, subtracted the sum from 306, and then divided by 3 to isolate the value of *N*. While the student in this study was not able to talk about why he did what he did, this researcher has observed that many students can provide a rationale when using this approach. For this student, having a conversation about the two systematic guess and check approaches and then his approach, might help him and others in a class articulate why the steps he took make sense.

The point is not to teach students to use particular strategies such as guess and check or unwind, but to think about how having students articulate their thinking and share it with others provides opportunities to develop students' ability to think algebraically while they work with approaches that are sensible to them. It was through the process of making their intuitive reasoning explicit, as students did in their written work and interviews, that the ways students made sense of the structural constraints of the problems became apparent. NCTM (2000) suggests that in algebra students' facility with symbol manipulation can be enhanced if it is based on extensive experience with quantities in contexts through which students develop an initial understanding of the meanings and uses of variables and an ability to associate symbolic expressions with problem contexts. By providing problem settings like as the ones presented, there are opportunities to have conversations that will support the development of algebraic thinking and students' transition to algebra.

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