Math 4710 HOMEWORK #3 Name:

Due 2/10/2023, 10:30 a.m., before start of the class

Solve the following problems and staple your solutions to this cover sheet.

In the following problems, we establish the results needed for d'Alembert's solution of the wave equation and then derive the d'Alembert solution. We did solve almost all of them in the BVP course.

1. Suppose (piecewise) continuous function h is a p-periodic (periodic with period p). Show that $\displaystyle\int_a^{a+p} h(x)\,dx = \int_0^p h(x)\,dx$ for any constant a.
   Hint: Start with $\int_a^{a+p} h(x)\,dx = \int_a^p h(x)\,dx + \int_p^{a+p} h(x)\,dx$ and use the substitution $u = x - p$ in the second integral.

2. (a) Show that if h is an odd (piecewise) continuous function, then $H(x) = \dfrac{1}{c}\displaystyle\int_0^x h(\xi)\,d\xi$ is an even function.
   (b) Show that if h is an odd, p-periodic (piecewise) continuous function, then $H(x) = \dfrac{1}{c}\displaystyle\int_0^x h(\xi)\,d\xi$ is a p-periodic function.

   Hints: To prove periodicity, show $H(x+p) = \frac{1}{c}\int_0^x h(\xi)\,d\xi + \frac{1}{c}\int_x^{x+p} h(\xi)\,d\xi$ and apply the last problem to the second integral twice; first with $a = x$ and then $a = -\frac{p}{2}$. Use the fact that for any odd function f, $\int_{-b}^{b} f(x)\,dx = 0$.

3. (a) Show that if h is an even (piecewise) continuous function, then $H(x) = \dfrac{1}{c}\displaystyle\int_0^x h(\xi)\,d\xi$ is an odd function.
   (b) Show that if h is an even p-periodic (piecewise) continuous function, with $\displaystyle\int_0^p h(x)\,dx = 0$, then $H(x) = \dfrac{1}{c}\displaystyle\int_0^x h(\xi)\,d\xi$ is a p-periodic function.

   Hint: To prove periodicity, show $H(x+p) = \frac{1}{c}\int_0^x h(\xi)\,d\xi + \frac{1}{c}\int_x^{x+p} h(\xi)\,d\xi$ and apply the result of problem #1 to the second integral.

4. (a) Suppose h is an even differentiable function. Show that h' is an odd function.
   (b) Suppose h is an odd differentiable function. Show that h' is an even function.
   (c) Suppose h is a p-periodic even differentiable function. Show that h' is a p-periodic function.

   Hint: Take the implicit differentiation of $h(-x) = h(x)$, $h(-x) = -h(x)$, or $h(x+p) = h(x)$, for each part, respectively.

5. Suppose that $\phi(z)$ and $\psi(z)$ are twice differentiable functions. Show that the function $u(x, t) = \phi(x + ct) + \psi(x - ct)$ satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$.
   Hint: You solved this problem in the BVP course.

6. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty\,,\ t > 0 \\ u(x,\,0) &= f(x), & -\infty < x < \infty \\ \frac{\partial u}{\partial t}(x,\,0) &= g(x), & -\infty < x < \infty \end{aligned}$$

using the d'Alembert's solution.
Hint: See the lecture notes. $u(x,\,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi\,.$

7. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, & 0 < x < \infty\,,\ t > 0 \\ u(0,\,t) &= 0, & t > 0 \\ u(x,\,0) &= f(x), & 0 < x < \infty \\ \frac{\partial u}{\partial t}(x,\,0) &= g(x), & 0 < x < \infty \end{aligned}$$

using the d'Alembert's solution.
Hints: See the lecture notes. The above results might be helpful. $u(x,\,t) = \frac{1}{2}(f_{o}(x+ct) + f_{o}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{o}(\xi)\,d\xi\,.$

8. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, & 0 < x < L\,,\ t > 0 \\ u(0,\,t) &= 0, & t > 0 \\ u(L,\,t) &= 0, & t > 0 \\ u(x,\,0) &= f(x), & 0 < x < L \\ \frac{\partial u}{\partial t}(x,\,0) &= g(x), & 0 < x < L \end{aligned}$$

using the d'Alembert's solution.
Hints: See class notes. For extension of g, use the result of the above problems. $u(x,\,t) = \frac{1}{2}(\bar{f}_{o}(x+ct) + \bar{f}_{o}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{o}(\xi)\,d\xi\,.$

9. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, & 0 < x < L\,,\ t > 0 \\ \frac{\partial u}{\partial x}(0,\,t) &= 0, & t > 0 \\ \frac{\partial u}{\partial x}(L,\,t) &= 0, & t > 0 \\ u(x,\,0) &= f(x), & 0 < x < L \\ \frac{\partial u}{\partial t}(x,\,0) &= g(x), & 0 < x < L \end{aligned}$$

using the d'Alembert's solution, assuming $\displaystyle\int_0^L g(x)\,dx = 0.$
Hints: See class notes. Use the result of the above problems.You will need to show that $\int_0^p \bar{g}_{e}(x)\,dx = 0$ for $p = 2L$. From the above problems, $\int_0^{2L} \bar{g}_{e}(x)\,dx = \int_{-L}^{L} \bar{g}_{e}(x)\,dx = 2\int_0^L g(x)\,dx = 0$. $u(x,\,t) = \frac{1}{2}(\bar{f}_{e}(x+ct) + \bar{f}_{e}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{e}(\xi)\,d\xi\,.$

10. Free points!