Math 3710 HOMEWORK #10 Name:

Due 11/18/2022, 12:30, before start of the class.

Solve the following problems and staple your solutions to this cover sheet.

Solve the following problem wave equation problem using the d'Alembert's solution in the following three steps.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, & 0 < x < L\,,\ t > 0 \\ \frac{\partial u}{\partial x}(0,\, t) = \frac{\partial u}{\partial x}(L,\, t) &= 0, & t > 0 \\ u(x,\, 0) &= f(x), & 0 < x < L \\ \frac{\partial u}{\partial t}(x,\, 0) &= 0, & 0 < x < L \end{aligned}$$

1. Starting with $u(x,\, t) = \phi(x + ct) + \psi(x - ct)$, show $\phi(x) = \frac{1}{2}[\, f(x) + k]$, $\psi(x) = \frac{1}{2}[\, f(x) - k]$, and then $u(x,\, t) = \frac{1}{2}[\, f(x + ct) + f(x - ct)]$, assuming $0 < x \pm ct < L$.
   Hint: This is just like a problem from the last homework, but shorter.

2. Suppose $u(x,\, t) = \frac{1}{2}[\, \tilde{f}(x + ct) + \tilde{f}(x - ct)]$ where $\tilde{f}$ is the extensions of f to the entire real number line. Show $\left(\tilde{f}\right)'$ is an odd, $2L$-periodic function.
   Hint: Apply the boundary conditions. This is just like a problem from the last homework, except that you will have $\left(\tilde{f}\right)'$ in place of $\tilde{f}$.

3. Show $u(x,\, t) = \frac{1}{2}[\, \bar{f}_e(x + ct) + \bar{f}_e(x - ct)]$.
   Hints: Use the results from HW 2. (If f is p-periodic, then f' is also p-periodic. If f is an even differentiable function, then f' is an odd function.) Here we have $\bar{f}_e(x)$ in place of $f(x)$.

4. Sec 5.3 #5
   Note: You may use Review, Identities, Theorems, Formulas and Tables handout. Hints: $\phi_1(x) = 1$ and $\phi_n(x) = \cos\frac{(n-1)\pi x}{L}$ for $n = \mathbf{2}, 3, \cdots$. See class notes.

5. Sec 5.3 #7
   Note: You may use Review, Identities, Theorems, Formulas and Tables handout. See class notes.

6. Prove that the eigenfunctions of the Regular Sturm-Liouville EVP, corresponding to different eigenvalues, are orthogonal with the weight function $\sigma(x)$.
   Note: Show part 5 of the S-L theorem. Hint: See class notes or section 5.5.

7. Derive the Rayleigh quotient for the Regular Sturm-Liouville EVP.
   Note: Show part 5 of the S-L theorem. Hint: See class notes or section 5.6.

8. Sec 5.3 #8
   Hints: Apply the Rayleigh quotient. Use the fact that if on $(a,\, b)$ function f is continuous, nonnegative, and $f \not\equiv 0$, then $\int_a^b f(x)\, dx > 0$, for $f = \phi^2$ and $f(x) = x^2\phi^2(x)$. See class notes, for similar problems.

9. Consider the Regular Sturm-Liouville Problem and Theorem. Show that if $f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$ for $a < x < b$, then $a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\, dx}{\int_a^b \phi_n^2(x)\sigma(x)\, dx}$.
   Hint: Let m be a positive integer. Multiply both sides by $\phi_m(x)\sigma(x)$ and integrate. You may change the order of summation and integration. Use the orthogonality condition: part 5 of S-L theorem.

10. Free points!