Probability and Statistics — A. Ghoreishi

Sample Final Exam Questions — 55 problems

Chapter 5 & Sections 6.1-6.6, 8.7, 8.9, 9.1, 9.2, 9.6, 9.7, 10.4, 10.5, 10.9, 11.1-11.10, 13.1-13.4

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1. A bag contains 8 chips; 3 red, 4 green and 1 white. Suppose three chips are chosen in random. Let Y_1 be the number of red chips, Y_2 the number of green chips, and Y_3 the number of white chips in the sample. Find The joint probability function of Y_1 and Y_2; $P(Y_1 = y_1,\, Y_2 = y_2)$. Find $E(Y_3)$ and $Cov(Y_1,\, Y_2)$.

2. Assume that X, Y, and Z are three random variables, with $E(X) = 2$, $E(Y) = -1$, $E(Z) = 3$, $V(X) = 3$, $V(Y) = 5$, $V(Z) = 7$, $Cov(X,\, Y) = 2$, $Cov(X,\, Z) = 4$, and $Cov(Y,\, Z) = -1$. Find $E(5X + Y - 3Z)$, $V(2X - Y)$, and $Cov(2X - Y,\, 5X + Y - 3Z)$.

3. An urn contains 9 marbles; 4 red, 3 green and 2 white. Suppose three marbles are chosen in random. Let Y_1 be the number of red marbles in the sample and let Y_2 be the number of green marbles in the sample. Find the following.
   (a) The joint probability function of Y_1 and Y_2. $P(Y_1 = 2,\, Y_2 = 1)$. The joint (cumulative) distribution function at $(Y_1,\, Y_2) = (2,\, 1)$; $F(2,\, 1)$.
   (b) The marginal probability functions of Y_1 and Y_2.
   (c) The conditional probability function values $P(Y_1 = 2\,|Y_2 = 1)$ and $P(Y_2 = 1\,|Y_1 = 2)$. The conditional distribution function value $P(Y_1 \leq 2\,|Y_2 = 1)$.
   (d) $E(Y_1)$, $E(Y_2)$, $E(Y_1^2)$, $V(Y_1)$ and $Cov(Y_1, Y_2)$.
   (e) $Cov(2Y_1 + Y_2,\, Y_1 - 2Y_2)$, $E(Y_1\,|Y_2 = 1)$, $E(Y_1^2\,|Y_2 = 1)$ and $V(Y_1\,|Y_2 = 1)$.

4. Two components of a minicomputer have the following joint probability density function for their useful lifetimes Y_1 and Y_2:

$$f(y_1,\, y_2) = \begin{cases} y_1 e^{-y_1(1+y_2)} & ,\ y_1 \geq 0 \text{ and } y_2 \geq 0 \\ 0 & ,\ \text{otherwise} \end{cases}$$

   What is the probability that the lifetime Y_1 of the first component exceeds 3? What are the marginal probability density functions of Y_1 and Y_2? Are Y_1 and Y_2 independent? What is the probability that the lifetime of at least one component exceeds 3?

5. Suppose random variables Y_1 and Y_2 are jointly continuous with the joint probability density function $f(y_1,\, y_2) = c\,e^{-y_1 - y_2}$, $0 \leq y_1 \leq y_2 < \infty$, where c is a constant. Find the value of c. Find $f_2(y_2)$, $f(y_1|\, y_2)$ and $E(Y_1|\, Y_2 = 2)$.

6. Let $f(y_1,\, y_2) = 2e^{-y_1 - y_2}$, $0 \leq y_1 \leq y_2 < \infty$, be the joint probability density function of Y_1 and Y_2. Find $f_1(y_1)$, $f_2(y_2)$. Are Y_1 and Y_2 independent?

7. Let $f(y_1,\, y_2) = y_1 + y_2$, $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$, be the joint probability density function of Y_1 and Y_2. Find $E(Y_1)$, $E(Y_2)$, $E(Y_1 Y_2)$, $V(Y_1)$ and $Cov(Y_1,\, Y_2)$.

8. Suppose Y_1 and Y_2 are jointly continuous random variables with joint probability density function $f(y_1,\, y_2)$. Prove that $Cov(Y_1,\, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$. Use this formula to show that $Cov(Y_1 + Y_2,\, Y_1 - Y_2) = V(Y_1) - V(Y_2)$.

9. Let Y_1 be the number of ones and Y_2 the number of twos and threes when a pair of fair dice are rolled. Find the joint probability function of Y_1 and Y_2. What is the probability $P(Y_1 = 1, Y_2 = 1)$? Find the variances of Y_1 and Y_2 and the covariance of Y_1 and Y_2.

10. In a large population of voters, 50% favor candidate A, 40% favor candidate B, and 10% favor candidate C. A random sample of size 10 is taken. State the joint probability function $P(X = x,\ Y = y,\ Z = z)$, where X, Y, and Z are the number of people favoring candidate A, B, and C, respectively. What is the probability that in our sample the same proportions as in the whole population would be found? What is the probability that 6 people in our sample favor candidate A if exactly 2 people in the sample favor candidate C?

11. Let Y_1 and Y_2 be continuous random variables with joint probability density function

$$f(y_1,\ y_2) = \begin{cases} \frac{1}{8}(6 - y_1 - y_2) & ,\ 0 < y_1 < 2 \text{ and } 2 < y_2 < 4 \\ 0 & ,\ \text{otherwise.} \end{cases}$$

Find marginal pdf $f_1(y_1)$, conditional pdf $f(y_2|y_1)$, $E(Y_2^2\,|\,Y_1 = 1)$, and $E(Y_1)$.

12. Let $f(y_1,\ y_2) = \frac{1}{8}$, $0 \le y_2 \le 4$, $y_2 \le y_1 \le y_2 + 2$, be the joint probability density function of Y_1 and Y_2. Find $f(y_1\,|y_2)$, $E(Y_1\,|Y_2)$ and $V(Y_1\,|Y_2)$.

13. Suppose random variable Y has an exponential distribution with mean β. Furthermore assume that β itself is a random variable with a uniform distribution over the interval $(1,\ 5)$. Find $E(Y)$ and $V(Y)$.

14. Two main companies compete for the same government contracts and receive the majority of them. Let X and Y represent the number of contracts each obtain in a year. What would you expect the sign of $Cov(X,\ Y)$ to be and why?

15. Let $Y_1,\ Y_2,\ \cdots,\ Y_n$ be independent random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$, for $i = 1,\ 2,\ \cdots,\ n$. Prove that $E(\overline{Y}) = \mu$ and $V(\overline{Y}) = \frac{\sigma^2}{n}$.

16. Let Y represent the number of successes in a binomial experiment with n trials and the probability of success in each trial p. Suppose the probability of success is also a random variable with uniform distribution over the interval $0.2 \le p \le 0.8$. Find $E(Y)$ and $V(Y)$.

17. Let X have a uniform distribution over the interval $(0,\ 2)$ and the the conditional distribution of Y, given $X = x$, be uniform over the interval $(0,\ x^2)$. Find the joint probability density function of X and Y, $f(x,\ y)$. Find the marginal probability function of Y, $f_2(y)$. Find $E(X|Y = y)$.

18. Assume random variable X is uniformly distribution over the interval $(0,\ 5)$. Suppose the conditional distribution of random vaiable Y, given $X = x$, is uniform over the interval $(0,\ x)$. Find the joint probability density function of X and Y, $f(x,\ y)$. Find the marginal probability function of Y, $f_2(y)$.

19. Let Y_1 and Y_2 denote random variables with the joint probability density function $f(y_1,\ y_2)$ and marginal probability density functions $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that

$$E(Y_1) = E[E(Y_1|Y_2)]$$

where, on the right-hand side, the inside expectation is with respect to the conditional distribution of Y_1 given Y_2, and the outside expectation is with respect to the distribution of Y_2.

20. Suppose U is a uniformly distributed random variable on the interval $[-1, 1]$. Find the probability density function of $Y = U^2$.

21. Suppose U is a uniformly distributed random variable on the interval $[0, 1]$. Find the probability density function of $Y = U^2$.

22. Suppose that Z is a normally distributed random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$. Show that Z^2 has χ^2 distribution with $\nu = 1$ degree of freedom.

23. Let Y_1 and Y_2 be continuous random variables with joint probability density function

$$f(y_1, y_2) = \begin{cases} \frac{1}{8}(6 - y_1 - y_2) & , 0 < y_1 < 2 \text{ and } 2 < y_2 < 4 \\ 0 & , \text{otherwise.} \end{cases}$$

Find the probability density function for the random variable $U = Y_1 + Y_2 + 2$.

24. Let Y be a random variable with probability density function

$$f(y) = \begin{cases} 6y(1-y) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Find the probability density function for the random variable $U = 3Y - 2$.

25. Suppose that Z is a normally distributed random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$. Show that Z^2 has χ^2 distribution with $\nu = 1$ degree of freedom.

26. Let $Y_1, \cdots, Y_n$ be independent, uniformly distributed random variables on the interval $(0, \theta)$. Find the probability density function of the 1st-order statistic $Y_{(1)} = \text{Min}\{Y_1, \cdots, Y_n\}$ and its mean.

27. Let $Y_1, \cdots, Y_n$ be independent, uniformly distributed random variables on the interval $(0, \theta)$. Find the probability density function of the nth-order statistic $Y_{(n)} = \text{Max}\{Y_1, \cdots, Y_n\}$ and its variance.

28. Let $Y_1, \cdots, Y_4$ be four independent, uniformly distributed random variables on the interval $(0, 1)$. Find the probability density function of the 3rd-order statistic $Y_{(3)}$ and $P(\frac{1}{3} < Y_{(3)} < \frac{2}{3})$.

29. Suppose a population is normally distributed with mean μ and standard deviation $\sigma = 1.00$.

    (a) How many observations should be included in a sample if we wish the sample mean $\overline{Y}$ to be within 0.307 of μ with probability 0.95.

    (b) For the sample size obtained in (a), find the numbers c and d such that $P(c \leq S^2 \leq d) = 0.95$ where S^2 is the variance of the sample.

30. Suppose we know that the unemployment rate has been about 8%. However, we wish to update our estimate in order to make an important decision about the national economic policy. Accordingly, let us say we wish to be 99% confident that the new estimate of p is within 0.001 of the true p. What is the sample size needed to achieve this?

31. Suppose that a college of 3000 students is interested in assessing student support for a new form for teacher evaluation. To estimate the proportion p in favor of the new form, how large a sample is required so that with 95% confidence the maximum error of the estimate of p is 0.03.

32. Suppose gasoline consumption of an experimental engine measured under a standard test is approximately normally distributed with standard deviation σ. The standard deviation of the gasoline consumption in 16 test runs was $S = 2.2$. Construct a 99% confidence interval for σ^2.

33. Suppose we want to study the average number of minutes of commercials in an one-hour period on television. A random sample of 36 different one-hour periods revealed a standard deviation of 3 minutes of commercials. If we wanted to test $H_0 : \mu = 15$ versus $H_a : \mu = 16$, what would be the β for this test? Use $\alpha = 0.05$.

34. Suppose we want to study the result of a new teaching method in test scores. A preliminary study shows that the variance of scores on a standardized test for students in the new teaching method is 12. If we want to detect a three point difference in the test scores: $H_0 : \mu = 72$ versus $H_a : \mu = 75$ with $\alpha = \beta = 0.05$, how many students should be taught using the new teaching method?

35. Let Y equal the number of pounds of butterfat produced by a Holstein cow during the 305-day milking period following the birth of a calf. We shall test the null hypothesis $H_0 : \sigma^2 = 140^2$ against the alternative hypothesis $H_a : \sigma^2 > 140^2$.

   (a) Give the test statistic and a rejection region that has a significance level of $\alpha = 0.05$, assuming that there are $n = 25$ observations.

   (b) Calculate the value of test statistic and give your conclusion using the following 25 observations of Y.

425	710	661	664	732	714	934	761	744	653	725	657	421
573	535	602	537	405	874	791	721	849	567	468	975	

36. In May the fill weights of 6-pound boxes of laundry soap had a mean of 6.13 pounds with a standard deviation of 0.095. The goal was to decrease the standard deviation. The company decided to adjust the filling machines and then test $H_0 : \sigma^2 = 0.095^2$ versus $H_a : \sigma^2 < 0.095^2$. In June a random sample of size 20 yielded sample mean of 6.10 and sample standard deviation of 0.065.

   (a) At an $\alpha = 0.05$ significance level, was the company successful?

   (b) What is the approximate p-value of your test?

37. Let Y_1 and Y_2 equal the times in days required for maturation of Guardiola seeds from narrow-leaved and broad leaved parents. Assume that the distributions of Y_1 and Y_2 are normal and independent with variances σ_1^2 and σ_2^2, respectively. Test $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$ versus $H_a : \frac{\sigma_1^2}{\sigma_2^2} > 1$ is a sample of size of 13 from Y_1 yielded sample variance of $s_1^2 = 9.88$ and a sample of size of 9 from Y_2 yielded sample variance of $s_2^2 = 4.08$. Use $\alpha = 0.05$.

38. Let Y_1 equal the weight in grams of a Low Fat Strawberry Kudo and Y_2 the weight in grams of a Low Fat Blueberry Kudo. Assume that the distributions of Y_1 and Y_2 are normal and independent with variances σ_1^2 and σ_2^2, respectively. Use the following 9 observations of Y_1:

   21.7 21.0 21.2 20.7 20.4 21.9 20.2 21.6 20.6

   and 13 observations of Y_2:

   21.5 20.5 20.3 21.6 21.7 21.3 23 21.3 18.9 20.0 20.4 20.8 20.3

   to test $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$ versus $H_a : \frac{\sigma_1^2}{\sigma_2^2} \neq 1$. Use $\alpha = 0.05$.

39. Let $Y_1, \cdots, Y_n$ be a random sample from a Poisson distribution with mean λ. Show that $\hat{\lambda}_1 = Y_1$ and $\hat{\lambda}_2 = \overline{Y}$ are two unbiased estimators of λ. Calculate the efficiency of $\hat{\lambda}_1$ relative to $\hat{\lambda}_2$.

40. Let $Y_1, \cdots, Y_n$ be a random sample from a uniform distribution over the interval $(0, \theta)$. Show that $\hat{\theta}_1 = \frac{2}{n}\sum_{i=1}^{n} Y_i$ and $\hat{\theta}_2 = \frac{n+1}{n} Y_{(n)}$ are both unbiased estimators of θ. Find the relative efficiency $\text{eff}(\hat{\theta}_1, \hat{\theta}_2)$.

41. Let $Y_1, \cdots, Y_n$ be a random sample of size n from a distribution with probability density function $f(y) = \theta\, y^{\theta-1}$, $0 < y < 1$, $\theta > 0$. Find the method of moment estimator of θ.

42. Let $Y_1, \cdots, Y_n$ be a random sample from a geometric distribution with the probability function $p(y) = p(1-p)^{y-1}$, $y = 1, \cdots, n$, $0 < p < 1$. Find the maximum-likelihood estimator of p.

43. Let $Y_1, \cdots, Y_n$ denote the outcomes of a series of independent trials, where

$$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}, \text{ for } i = 1, \cdots, n \text{ and } 0 < p < 1.$$

Let $Y = Y_1 + \cdots + Y_n$. Show that $\hat{p}_1 = Y_1$ and $\hat{p}_2 = \frac{Y}{n}$ are unbiased estimators of p. Find the relative efficiency $\text{eff}(\hat{p}_1, \hat{p}_2)$. Hint: Each Y_i has a binomial distribution with sample size 1 and probability of success p.

44. A random sample of n observations, $Y_1, \cdots, Y_n$, is selected from a population where Y_i, for $i = 1, \cdots, n$, possesses a gamma distribution with mean $\mu = \alpha\beta$ and variance $\sigma^2 = \alpha\beta^2$. Find the method of moments estimators of the unknown parameters α and β.

45. Suppose that $Y_1, \cdots, Y_n$ be a random sample from a population with density function

$$f(y) = \begin{cases} e^{-y+\theta} & , y \geq \theta \\ 0 & , \text{otherwise} \end{cases}.$$

Find an estimator $\hat{\theta}$ for θ by the method of moments. Is this estimator unbiased?

46. Suppose that $Y_1, \cdots, Y_n$ denote a sample from a population with density function

$$f(y) = \begin{cases} (\theta+1)y^{\theta} & , 0 \leq y \leq 1 \\ 0 & , \text{otherwise} \end{cases},$$

where $\theta > -1$. Find the maximum-likelihood estimator of θ.

47. Let Y_1 and Y_2 be a random sample from a population with uniform density function over the interval $(0, \theta)$. Show that $\hat{\theta}_1 = 3\,Y_1^2$ and $\hat{\theta}_2 = 2\,Y_1^2 + Y_2^2$ are two unbiased estimators of θ^2. Find the relative efficiency $\text{eff}(\hat{\theta}_1, \hat{\theta}_2)$. Notes: $V(Y) = E(Y^2) - [E(Y)]^2$ and $Cov(Y_1^2, Y_2^2) = 0$.

48. By late 1971, all cigarette packs had to be labeled with the words, "Warning: The Surgeon General Has Determined That Smoking Is Dangerous To Your Health". The case against smoking rested heavily on statistical evidence. The following tables shows the annual cigarette consumption and the corresponding mortality rate due to the coronary hear disease (CHD) for 21 countries. Does the data support the suspicion that smoking contributes to CHD mortality? Test $H_0 : \beta_1 = 0$ versus $H_a : \beta_1 > 0$ at the $\alpha = 0.05$ level of significance.

Country	Cigarette Consumption per Adult per Year, x	CHD Mortality per 100,000 (ages 35-64), y
United States	3900	256.9
Canada	3350	211.6
Australia	3220	238.1
New Zealand	3220	211.8
United Kingdom	2790	194.1
Switzerland	2780	124.5
Ireland	2770	187.3
Iceland	2290	110.5
Finland	2160	233.1
West Germany	1890	150.3
Netherlands	1810	124.7
Greece	1800	41.2
Austria	1770	182.1
Belgium	1700	118.1
Mexico	1680	31.9
Italy	1510	114.3
Denmark	1500	144.9
France	1410	59.7
Sweden	1270	126.9
Spain	1200	43.9
Norway	1090	136.3

49. For many firms, expenses are linear functions of sales. That appears to be the case for ACME Manufacturing, a company that markets industrial gases. The following table shows Acme's annual sale and expenses for the years 1985 through 1992. For the model $E(Y) = \beta_0 + \beta_1 x$ we can calculate the estimators $\hat{\beta}_0 = 26.73$ and $\hat{\beta}_1 = 0.22$. Find a 95% confidence interval for β_1.

Year	Sales (in thousands), x	Expenses (in thousands), y
1985	1756	407
1986	1942	466
1987	2132	489
1988	2431	545
1989	2642	610
1990	2895	659
1991	2931	686
1992	3217	724

50. Consider the last problem. Suppose the company's management believes that sales will stabilize over the next several years at the $3,500,000 level. Construct a 95% confidence interval for $E(Y)$ when $x = x^* = 3500$.

51. Some baseball fans believe that the number of home runs a team hits is markedly affected by the altitude of the club's home park. The following table shows the altitudes of the American League ballparks and the number of home runs that each team hit during a recent season. Calculate the sample correlation coefficient, r. Test whether home run frequency and home park altitude are independent. Use $\alpha = 0.05$.

Club	Altitude, x	Number of Home Runs, y
Cleveland	660	138
Milwaukee	635	81
Detroit	585	135
New York	55	90
Boston	21	120
Baltimore	20	84
Minnesota	815	106
Kansas City	750	57
Chicago	595	109
Texas	435	74
California	340	61
Oakland	25	120

52. Over the past 25 years computers have steadily decreased in size as they have grown in power. The ability to have more computing power in a 4-pound laptop than in a mainframe of the 1970's is a result of engineers squeezing more and more transistors onto silicon chips. The rate at which this miniaturization occurs is known as Moore's law, after Gordon Moore, one of the founders of the Intel corporation. His prediction, first articulated in 1965, was that the number of transistors per chip will double every 18 months. The following table lists some of growth benchmarks– namely, the number of transistors per chip–associated with the Intel chips marketed over the 20-year period from 1975 through 1995. Bases on these figures, the appropriate model for this data is $E(Y) = \alpha_0 \, e^{\alpha_1 x}$. Linearize this model and estimate the parameters by the method of least squares. Find the doubling time for $E(Y)$.

Chip	Year	Years after 1975, x	Transistors per Chip, y
8080	1975	0	4,500
8086	1978	3	29000
80286	1982	7	90,000
80386	1985	10	229,000
80486	1989	14	1,200,000
Pentium	1993	18	3,100,000
Pentium Pro	1995	20	5,500,000

53. A window that is manufactured for an automobile has five studs to attach it. A company that manufactures these windows perform "pull-out tests" to determine the force needed to pull a stud out of the window. Let Y_i, $i = 1, \cdots, 5$, equal the force required at position i and assume that they are normally distributed with equal variance and mean μ_i, respectively. Test $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ versus H_a : Not H_0 at the level $\alpha = 0.01$ using the following data.

Y_1	Y_2	Y_3	Y_4	Y_5
92	100	143	142	147
90	108	149	155	144
87	98	138	119	160
105	110	136	134	149
86	114	139	133	152
83	97	120	146	131
102	94	145	152	134

54. Four types of shipping containers were compared with respect to compression strength. The data on 6 boxes of each type are collected as shown below. Is there a significant difference between theses boxes? Use $\alpha = 0.05$. What is the p-value for this test?

Type of Box	Compression Strength					
A	655.5	788.8	734.3	721.4	679.1	699.4
B	789.2	772.5	786.9	686.1	732.1	774.8
C	737.1	639.0	696.3	671.7	717.2	727.1
D	535.1	628.7	542.4	559.0	586.9	520.0

55. A biologist wished to study the effects of ethanol on sleep time. A sample of 20 rats, matched for age and other characteristics, was selected, and each rat was given an oral injection having a particular concentration of ethanol per body weight. The rapid eye movement (REM) sleep time for each rat was then recorded for a 24-hour period, with the following results:

Treatment	Sleep Time					$Y_{i\bullet}$	$\overline{Y}_{i\bullet}$
0 (control)	88.6	73.2	91.4	68.0	75.2	396.4	79.28
1 g/kg	63.0	53.9	69.2	50.1	71.5	307.7	61.54
2 g/kg	44.9	59.5	40.2	56.3	38.7	239.6	47.92
4 g/kg	31.0	39.6	45.3	25.2	22.7	163.8	32.76
						$Y_{\bullet\bullet} = 1107.5$	$\overline{Y}_{\bullet\bullet} = 55.375$

Does the data indicate that the true average REM sleep time depends on the concentration of ethanol? Four types of shipping containers were compared with respect to compression strength. The data on 6 boxes of each type are collected as shown below. Is there a significant difference between theses boxes? Use $\alpha = 0.05$. What is the p-value for this test?