Probability and Statistics — A. Ghoreishi

Sample Exam II Questions — 26 problems

Sections 6.4-6.6, 8.7, 8.9, 9.1, 9.2, 9.6, 9.7, 10.4, 10.5, 10.9

Math 3420, ID #1967601, Spring 2002

1. Let Y be a random variable with probability density function

$$f(y) = \begin{cases} 6y(1-y) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Find the probability density function for the random variable $U = 3Y - 2$.

2. Suppose U is a uniformly distributed random variable on the interval $[0,\, 1]$. Find the probability density function of $Y = U^2$.

3. Suppose that Z is a normally distributed random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$. Show that Z^2 has χ^2 distribution with $\nu = 1$ degree of freedom.

4. Let Y_1 and Y_2 be continuous random variables with joint probability density function

$$f(y_1,\, y_2) = \begin{cases} \frac{1}{8}\,(6 - y_1 - y_2) & ,\, 0 < y_1 < 2 \text{ and } 2 < y_2 < 4 \\ 0 & ,\, \text{otherwise.} \end{cases}$$

Find the probability density function for the random variable $U = Y_1 + Y_2 + 2$.

5. Let $Y_1,\, \cdots,\, Y_n$ be independent, uniformly distributed random variables on the interval $(0,\, \theta)$. Find the probability density function of the 1st-order statistic $Y_{(1)} = \text{Min}\{Y_1,\, \cdots,\, Y_n\}$ and its mean.

6. Let $Y_1,\, \cdots,\, Y_n$ be independent, uniformly distributed random variables on the interval $(0,\, \theta)$. Find the probability density function of the nth-order statistic $Y_{(n)} = \text{Max}\{Y_1,\, \cdots,\, Y_n\}$ and its variance.

7. Let $Y_1,\, \cdots,\, Y_4$ be four independent, uniformly distributed random variables on the interval $(0,\, 1)$. Find the probability density function of the 3rd-order statistic $Y_{(3)}$ and $P(\frac{1}{3} < Y_{(3)} < \frac{2}{3})$.

8. Suppose a population is normally distributed with mean μ and standard deviation $\sigma = 1.00$.

   (a) How many observations should be included in a sample if we wish the sample mean $\overline{Y}$ to be within 0.307 of μ with probability 0.95.

   (b) For the sample size obtained in (a), find the numbers c and d such that $P(c \leq S^2 \leq d) = 0.95$ where S^2 is the variance of the sample.

9. Suppose we know that the unemployment rate has been about 8%. However, we wish to update our estimate in order to make an important decision about the national economic policy. Accordingly, let us say we wish to be 99% confident that the new estimate of p is within 0.001 of the true p. What is the sample size needed to achieve this?

10. Suppose that a college of 3000 students is interested in assessing student support for a new form for teacher evaluation. To estimate the proportion p in favor of the new form, how large a sample is required so that with 95% confidence the maximum error of the estimate of p is 0.03.

11. Suppose gasoline consumption of an experimental engine measured under a standard test is approximately normally distributed with standard deviation σ. The standard deviation of the gasoline consumption in 16 test runs was $S = 2.2$. Construct a 99% confidence interval for σ^2.

12. Suppose we want to study the average number of minutes of commercials in an one-hour period on television. A random sample of 36 different one-hour periods revealed a standard deviation of 3 minutes of commercials. If we wanted to test $H_0 : \mu = 15$ versus $H_a : \mu = 16$, what would be the β for this test? Use $\alpha = 0.05$.

13. Suppose we want to study the result of a new teaching method in test scores. A preliminary study shows that the variance of scores on a standardized test for students in the new teaching method is 12. If we want to detect a three point difference in the test scores: $H_0 : \mu = 72$ versus $H_a : \mu = 75$ with $\alpha = \beta = 0.05$, how many students should be taught using the new teaching method?

14. Let Y equal the number of pounds of butterfat produced by a Holstein cow during the 305-day milking period following the birth of a calf. We shall test the null hypothesis $H_0 : \sigma^2 = 140^2$ against the alternative hypothesis $H_a : \sigma^2 > 140^2$.

    (a) Give the test statistic and a rejection region that has a significance level of $\alpha = 0.05$, assuming that there are $n = 25$ observations.

    (b) Calculate the value of test statistic and give your conclusion using the following 25 observations of Y.

425	710	661	664	732	714	934	761	744	653	725	657	421
573	535	602	537	405	874	791	721	849	567	468	975	

15. In May the fill weights of 6-pound boxes of laundry soap had a mean of 6.13 pounds with a standard deviation of 0.095. The goal was to decrease the standard deviation. The company decided to adjust the filling machines and then test $H_0 : \sigma^2 = 0.095^2$ versus $H_a : \sigma^2 < 0.095^2$. In June a random sample of size 20 yielded sample mean of 6.10 and sample standard deviation of 0.065.

    (a) At an $\alpha = 0.05$ significance level, was the company successful?

    (b) What is the approximate p-value of your test?

16. Let Y_1 and Y_2 equal the times in days required for maturation of Guardiola seeds from narrow-leaved and broad leaved parents. Assume that the distributions of Y_1 and Y_2 are normal and independent with variances σ_1^2 and σ_2^2, respectively. Test $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$ versus $H_a : \frac{\sigma_1^2}{\sigma_2^2} > 1$ is a sample of size of 13 from Y_1 yielded sample variance of $s_1^2 = 9.88$ and a sample of size of 9 from Y_2 yielded sample variance of $s_2^2 = 4.08$. Use $\alpha = 0.05$.

17. Let Y_1 equal the weight in grams of a Low Fat Strawberry Kudo and Y_2 the weight in grams of a Low Fat Blueberry Kudo. Assume that the distributions of Y_1 and Y_2 are normal and independent with variances σ_1^2 and σ_2^2, respectively. Use the following 9 observations of Y_1:

    21.7 21.0 21.2 20.7 20.4 21.9 20.2 21.6 20.6

    and 13 observations of Y_2:

    21.5 20.5 20.3 21.6 21.7 21.3 23 21.3 18.9 20.0 20.4 20.8 20.3

    to test $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$ versus $H_a : \frac{\sigma_1^2}{\sigma_2^2} \neq 1$. Use $\alpha = 0.05$.

18. Let $Y_1, \cdots, Y_n$ be a random sample from a Poisson distribution with mean λ. Show that $\hat{\lambda}_1 = Y_1$ and $\hat{\lambda}_2 = \overline{Y}$ are two unbiased estimators of λ. Calculate the efficiency of $\hat{\lambda}_1$ relative to $\hat{\lambda}_2$.

19. Let $Y_1, \cdots, Y_n$ be a random sample from a uniform distribution over the interval $(0, \theta)$. Show that $\hat{\theta}_1 = \frac{2}{n}\sum_{i=1}^{n} Y_i$ and $\hat{\theta}_2 = \frac{n+1}{n} Y_{(n)}$ are both unbiased estimators of θ. Find the relative efficiency $\text{eff}(\hat{\theta}_1, \hat{\theta}_2)$.

20. Let $Y_1, \cdots, Y_n$ be a random sample of size n from a distribution with probability density function $f(y) = \theta\, y^{\theta-1}$, $0 < y < 1$, $\theta > 0$. Find the method of moment estimator of θ.

21. Let $Y_1, \cdots, Y_n$ be a random sample from a geometric distribution with the probability function $p(y) = p(1-p)^{y-1}$, $y = 1, \cdots, n$, $0 < p < 1$. Find the maximum-likelihood estimator of p.

22. Let $Y_1, \cdots, Y_n$ denote the outcomes of a series of independent trials, where

$$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}, \text{ for } i = 1, \cdots, n \text{ and } 0 < p < 1.$$

Let $Y = Y_1 + \cdots + Y_n$. Show that $\hat{p}_1 = Y_1$ and $\hat{p}_2 = \frac{Y}{n}$ are unbiased estimators of p. Find the relative efficiency $\text{eff}(\hat{p}_1, \hat{p}_2)$. Hint: Each Y_i has a binomial distribution with sample size 1 and probability of success p.

23. A random sample of n observations, $Y_1, \cdots, Y_n$, is selected from a population where Y_i, for $i = 1, \cdots, n$, possesses a gamma distribution with mean $\mu = \alpha\beta$ and variance $\sigma^2 = \alpha\beta^2$. Find the method of moments estimators of the unknown parameters α and β.

24. Suppose that $Y_1, \cdots, Y_n$ be a random sample from a population with density function

$$f(y) = \begin{cases} e^{-y+\theta} & , y \geq \theta \\ 0 & , \text{otherwise} \end{cases}.$$

Find an estimator $\hat{\theta}$ for θ by the method of moments. Is this estimator unbiased?

25. Suppose that $Y_1, \cdots, Y_n$ denote a sample from a population with density function

$$f(y) = \begin{cases} (\theta+1)y^{\theta} & , 0 \leq y \leq 1 \\ 0 & , \text{otherwise} \end{cases},$$

where $\theta > -1$. Find the maximum-likelihood estimator of θ.

26. Let Y_1 and Y_2 be a random sample from a population with uniform density function over the interval $(0, \theta)$. Show that $\hat{\theta}_1 = 3\,Y_1^2$ and $\hat{\theta}_2 = 2\,Y_1^2 + Y_2^2$ are two unbiased estimators of θ^2. Find the relative efficiency $\text{eff}(\hat{\theta}_1, \hat{\theta}_2)$. Notes: $V(Y) = E(Y^2) - [E(Y)]^2$ and $Cov(Y_1^2, Y_2^2) = 0$.