Probability and Statistics — A. Ghoreishi

Sample Exam I Questions — 23 problems

# Chapter 5 & Sections 6.1-6.3
## Math 3420, ID #1967601, Spring 2002

1. A bag contains 8 chips; 3 red, 4 green and 1 white. Suppose three chips are chosen in random. Let $Y_1$ be the number of red chips, $Y_2$ the number of green chips, and $Y_3$ the number of white chips in the sample. Find The joint probability function of $Y_1$ and $Y_2$; $P(Y_1 = y_1, Y_2 = y_2)$. Find $E(Y_3)$ and $Cov(Y_1, Y_2)$.

2. Assume that $X$, $Y$, and $Z$ are three random variables, with $E(X) = 2$, $E(Y) = -1$, $E(Z) = 3$, $V(X) = 3$, $V(Y) = 5$, $V(Z) = 7$, $Cov(X, Y) = 2$, $Cov(X, Z) = 4$, and $Cov(Y, Z) = -1$. Find $E(5X + Y - 3Z)$, $V(2X - Y)$, and $Cov(2X - Y, 5X + Y - 3Z)$.

3. An urn contains 9 marbles; 4 red, 3 green and 2 white. Suppose three marbles are chosen in random. Let $Y_1$ be the number of red marbles in the sample and let $Y_2$ be the number of green marbles in the sample. Find the following.

   (a) The joint probability function of $Y_1$ and $Y_2$. $P(Y_1 = 2, Y_2 = 1)$. The joint (cumulative) distribution function at $(Y_1, Y_2) = (2, 1)$; $F(2, 1)$.

   (b) The marginal probability functions of $Y_1$ and $Y_2$.

   (c) The conditional probability function values $P(Y_1 = 2 \,|Y_2 = 1)$ and $P(Y_2 = 1 \,|Y_1 = 2)$. The conditional distribution function value $P(Y_1 \leq 2 \,|Y_2 = 1)$.

   (d) $E(Y_1)$, $E(Y_2)$, $E(Y_1^2)$, $V(Y_1)$ and $Cov(Y_1, Y_2)$.

   (e) $Cov(2Y_1 + Y_2, Y_1 - 2Y_2)$, $E(Y_1 \,|Y_2 = 1)$, $E(Y_1^2 \,|Y_2 = 1)$ and $V(Y_1 \,|Y_2 = 1)$.

4. Two components of a minicomputer have the following joint probability density function for their useful lifetimes $Y_1$ and $Y_2$:

$$f(y_1, y_2) = \begin{cases} y_1 e^{-y_1(1+y_2)} & , y_1 \geq 0 \text{ and } y_2 \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

   What is the probability that the lifetime $Y_1$ of the first component exceeds 3? What are the marginal probability density functions of $Y_1$ and $Y_2$? Are $Y_1$ and $Y_2$ independent? What is the probability that the lifetime of at least one component exceeds 3?

5. Suppose random variables $Y_1$ and $Y_2$ are jointly continuous with the joint probability density function $f(y_1, y_2) = c\,e^{-y_1 - y_2}$, $0 \leq y_1 \leq y_2 < \infty$, where $c$ is a constant. Find the value of $c$. Find $f_2(y_2)$, $f(y_1| \, y_2)$ and $E(Y_1| \, Y_2 = 2)$.

6. Let $f(y_1, y_2) = 2e^{-y_1 - y_2}$, $0 \leq y_1 \leq y_2 < \infty$, be the joint probability density function of $Y_1$ and $Y_2$. Find $f_1(y_1)$, $f_2(y_2)$. Are $Y_1$ and $Y_2$ independent?

7. Let $f(y_1, y_2) = y_1 + y_2$, $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$, be the joint probability density function of $Y_1$ and $Y_2$. Find $E(Y_1)$, $E(Y_2)$, $E(Y_1 Y_2)$, $V(Y_1)$ and $Cov(Y_1, Y_2)$.

8. Suppose $Y_1$ and $Y_2$ are jointly continuous random variables with joint probability density function $f(y_1, y_2)$. Prove that $Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$. Use this formula to show that $Cov(Y_1 + Y_2, Y_1 - Y_2) = V(Y_1) - V(Y_2)$.

9. Let $Y_1$ be the number of ones and $Y_2$ the number of twos and threes when a pair of fair dice are rolled. Find the joint probability function of $Y_1$ and $Y_2$. What is the probability $P(Y_1 = 1, Y_2 = 1)$? Find the variances of $Y_1$ and $Y_2$ and the covariance of $Y_1$ and $Y_2$.

10. In a large population of voters, 50% favor candidate A, 40% favor candidate B, and 10% favor candidate C. A random sample of size 10 is taken. State the joint probability function $P(X = x, Y = y, Z = z)$, where $X$, $Y$, and $Z$ are the number of people favoring candidate A, B, and C, respectively. What is the probability that in our sample the same proportions as in the whole population would be found? What is the probability that 6 people in our sample favor candidate A if exactly 2 people in the sample favor candidate C?

11. Let $Y_1$ and $Y_2$ be continuous random variables with joint probability density function

$$f(y_1, y_2) = \begin{cases} \frac{1}{8}(6 - y_1 - y_2) & , 0 < y_1 < 2 \text{ and } 2 < y_2 < 4 \\ 0 & , \text{otherwise.} \end{cases}$$

Find marginal pdf $f_1(y_1)$, conditional pdf $f(y_2|y_1)$, $E(Y_2^2 \,|\, Y_1 = 1)$, and $E(Y_1)$.

12. Let $f(y_1, y_2) = \frac{1}{8}$, $0 \le y_2 \le 4$, $y_2 \le y_1 \le y_2 + 2$, be the joint probability density function of $Y_1$ and $Y_2$. Find $f(y_1\,|y_2)$, $E(Y_1\,|Y_2)$ and $V(Y_1\,|Y_2)$.

13. Suppose random variable $Y$ has an exponential distribution with mean $\beta$. Furthermore assume that $\beta$ itself is a random variable with a uniform distribution over the interval $(1, 5)$. Find $E(Y)$ and $V(Y)$.

14. Two main companies compete for the same government contracts and receive the majority of them. Let $X$ and $Y$ represent the number of contracts each obtain in a year. What would you expect the sign of $Cov(X, Y)$ to be and why?

15. Let $Y_1, Y_2, \cdots, Y_n$ be independent random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$, for $i = 1, 2, \cdots, n$. Prove that $E(\overline{Y}) = \mu$ and $V(\overline{Y}) = \frac{\sigma^2}{n}$.

16. Let $Y$ represent the number of successes in a binomial experiment with $n$ trials and the probability of success in each trial $p$. Suppose the probability of success is also a random variable with uniform distribution over the interval $0.2 \le p \le 0.8$. Find $E(Y)$ and $V(Y)$.

17. Let $X$ have a uniform distribution over the interval $(0, 2)$ and the the conditional distribution of $Y$, given $X = x$, be uniform over the interval $(0, x^2)$. Find the joint probability density function of $X$ and $Y$, $f(x, y)$. Find the marginal probability function of $Y$, $f_2(y)$. Find $E(X|Y = y)$.

18. Assume random variable $X$ is uniformly distribution over the interval $(0, 5)$. Suppose the conditional distribution of random vaiable $Y$, given $X = x$, is uniform over the interval $(0, x)$. Find the joint probability density function of $X$ and $Y$, $f(x, y)$. Find the marginal probability function of $Y$, $f_2(y)$.

19. Let $Y_1$ and $Y_2$ denote random variables with the joint probability density function $f(y_1, y_2)$ and marginal probability density functions $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that

$$E(Y_1) = E[E(Y_1|Y_2)]$$

where, on the right-hand side, the inside expectation is with respect to the conditional distribution of $Y_1$ given $Y_2$, and the outside expectation is with respect to the distribution of $Y_2$.

20. Let $Y$ be a random variable with probability density function

$$f(y) = \begin{cases} 6y(1-y) & \text{for } 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Find the probability density function for the random variable $U = 3Y - 2$.

21. Suppose $U$ is a uniformly distributed random variable on the interval $[-1, 1]$. Find the probability density function of $Y = U^2$.

22. Suppose that $Z$ is a normally distributed random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$. Show that $Z^2$ has $\chi^2$ distribution with $\nu = 1$ degree of freedom.

23. Let $Y_1$ and $Y_2$ be continuous random variables with joint probability density function

$$f(y_1, y_2) = \begin{cases} \frac{1}{8}(6 - y_1 - y_2) & , 0 < y_1 < 2 \text{ and } 2 < y_2 < 4 \\ 0 & , \text{otherwise.} \end{cases}$$

Find the probability density function for the random variable $U = Y_1 + Y_2 + 2$.