## Some Useful Facts for Math 3410

The following is a collection of facts, mostly from Calculus, which we will use in our course. You may want to look them up or prove them.

**Binomial Theorem**: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

**Geometric Series**: $\sum_{k=0}^{\infty} a\,r^k = \frac{a}{1-r}$, if $|r|<1$; $\sum_{k=n+1}^{\infty} a\,r^k = \frac{a\,r^{n+1}}{1-r}$, if $|r|<1$; $\sum_{k=0}^{n} a\,r^k = \frac{a(1-r^{n+1})}{1-r}$

For $|r|<1$, $\frac{d}{dr}\left(\sum_{k=0}^{\infty} a\,r^k\right) = \frac{d}{dr}\left(\frac{a}{1-r}\right) \Rightarrow \cdots \Rightarrow \sum_{k=1}^{\infty} ak\,r^{k-1} = \frac{a}{(1-r)^2}$

For $|r|<1$, $\frac{d}{dr}\left(\sum_{k=1}^{\infty} ak\,r^k\right) = \frac{d}{dr}\left[\frac{a}{(1-r)^2}\right] \Rightarrow \cdots \Rightarrow \sum_{k=2}^{\infty} ak(k-1)\,r^{k-2} = \frac{2a}{(1-r)^3}$

**Maclaurin Series**: The Maclaurin series for $f(x) = e^x$ and $f(x) = (1-x)^{-r}$ converge to their corresponding functions.

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all real numbers $x$ ; $(1-x)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} x^k$, if $|x|<1$.

Letting $y = k+r$, we get $\sum_{y=r}^{\infty} \binom{y-1}{r-1} x^{y-r} = (1-x)^{-r}$, if $|x|<1$.

For $|x|<1$, $\frac{d}{dx}\left[\sum_{k=0}^{\infty} \binom{r+k-1}{r-1} x^k\right] = \frac{d}{dx}(1-x)^{-r} \Rightarrow \cdots \Rightarrow \sum_{k=1}^{\infty} \binom{r+k-1}{r-1} k\,x^{k-1} = r(1-x)^{-r-1}$ and

$$\frac{d}{dx}\left[\sum_{k=1}^{\infty} \binom{r+k-1}{r-1} k\,x^{k-1}\right] = \frac{d}{dx}\left[r(1-x)^{-r-1}\right] \Rightarrow \cdots \Rightarrow \sum_{k=2}^{\infty} \binom{r+k-1}{r-1} k(k-1)\,x^{k-2} = r(r+1)(1-x)^{-r-2}$$

**Equivalent Forms of $\binom{N}{n}$**:

$$\frac{N}{n}\binom{N-1}{n-1} = \binom{N}{n}; \quad \frac{N(N-1)}{n(n-1)}\binom{N-2}{n-2} = \binom{N}{n}; \quad \sum_{y=0}^{r} \binom{r}{y}\binom{N-r}{n-y} = \binom{N}{n}$$

**Limits**:

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ ; $\lim_{n\to\infty} \binom{n}{y} p^y (1-p)^{n-y} = \frac{e^{-\lambda}\,\lambda^y}{y!}$, where $np = \lambda$ is a fixed constant

$\lim_{N\to\infty} \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \binom{n}{y} p^y (1-p)^{n-y}$, where $\frac{r}{N} = p$ is a fixed constant

**Integrals**:

$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ ; $\int_0^{\infty} y^{\alpha-1}\,e^{-y}\,dy = (\alpha-1)\int_0^{\infty} y^{\alpha-2}\,e^{-y}\,dy$