Math 3280 HOMEWORK #2 Name:

Due 1/17/2025, 11:30 a.m., before start of the class

Solve the following problems and staple your solutions to this cover sheet. (Computer outputs must be put in the appropriate place in the solution, not attached as an appendix. You may physically cut and paste the output in the problem or allow appropriate space in the printout to add your hand written work.)

1. Chap 1, Exer 1(a). Note: Find the general solution and hand draw several solution curves. The constant C can be a negative number.

2. Chap 1, Exer 1(a). Note: Solve using Mathematica and plot the direction field and several solution curves on the same coordinate system. See attached, your first HW, or your book for Mathematica commands.

3. Chap 1, Exer 1(e). Note: Find the general solution and use the ContourPlot of Mathematica to plot several solution curves. See attached, your first HW, or your book for Mathematica commands.

4. Chap 1, Exer 1(e). Note: Solve using Mathematica and plot the direction field and several solution curves on the same coordinate system. See attached, your first HW, or your book for commands.

5. Find the explicit solution of IVP $(2t - x) + (2x - t)\frac{dx}{dt} = 0\,,\ x(1) = 3\,.$

6. Chap 1, Exer 6. Note: Assume $k > 0$. Discuss the long term behavior of the solution.

7. Solve the following nonlinear planar decoupled system with the given initial conditions.

$$\begin{aligned} \frac{dx}{dt} &= x\,y \\ \frac{dy}{dt} &= 2y \end{aligned} \qquad (x(0),\, y(0)) = (1,\, 1)$$

   Hint: First solve the 2nd equation.

8. Chap 1, Exer 8. Hints: The initial conditions are $\sigma_A(0) = \sigma_B(0) = 0\,.$ Solve for σ_A and use it to solve for $\sigma_B\,.$ The mass is $\sigma_B\,V\,.$

9. Chap 1, Exer 10(a)(ii). Note: Solve the ODE to determine the values of t for which the solution $x(t)$ is defined.

10. Chap 1, Exer 10(b). Note: In the back the answer is for if this problem was in part a. To answer it correctly, consider the existence and uniqueness theorem discussed in class with $f(t,\, x) = 2\sqrt{x}$ and find all initial values $(t_0,\, x_0)$ that satisfy the hypothesis of that theorem.

Mathematica Commands

To plot the curve defined by $f(x, y) = k$ use

`ContourPlot[`$f(x,y)$`==`k`, {x,a,b}, {y,c,d}]`

where `a`, `b` and `c`, `d` are the lower and upper range values of x and y, respectively.
To plot the curves $f(x, y) = k_1\,,\ \cdots\,,\ f(x, y) = k_n$, on the same coordinate system, use

`ContourPlot[{`$f(x,y)$`==`$k_1\,,\ \cdots\,,\ f(x,y)$`==`$k_n$`}, {x,a,b}, {y,c,d}]` .

Consider ODE $x'(t) = f(t, x)$. To solve it use

`DSolve[x'[t]==`$f(t,x)$`, x[t], t]`

with x in function $f(t, x)$ typed in as `x[t]`. The last two terms indicate that x is the unknown function of t and the independent variable is t. To solve the IVP $x'(t) = f(t, x)$, $x(t_0) = x_0$ use

`DSolve[{x'[t]==`$f(t,x)$`, x[`t_0`]==`x_0`}, x[t], t]`

with x in function $f(t, x)$ typed in as `x[t]`. Use `NDSolve` if the IVP can't be solved exactly. In this case, replace the last `t`, which indicates the independent variable, with a range of values of it: `{t, a, b}`. The solution can be graphed using

`Plot[Evaluate[x[t] /.  %], {t,a,b}, PlotRange->{{a,b},{c,d}}]`

where `a`, `b` and `c`, `d` are the lower and upper range values of t and x, respectively. The symbol `/.` means replace and `%` recalls the last output, so `x[t] /.  %` means replace `x[t]` with the last output.

Consider ODE $x'(t) = f(t, x)$. Its direction field can be graphed using the `VectorPlot` command and `StreamPlot` shows the flow with embedded arrowheads. Do not use `StreamPlot` in place of graphing actual solutions.

`VectorPlot[{1,`$f(t,x)$`}, {t,a,b}, {x,c,d}], VectorScaling->Automatic]`

`StreamPlot[{1,`$f(t,x)$`}, {t,a,b}, {x,c,d}]`

with x in function $f(t, x)$ typed in as just `x`. In some situations, for better visualization, it might be necessary to draw all vectors with the same length. To do this try the following.

`VectorPlot[{1,`$f(t,x)$`}, {t,a,b}, {x,c,d}]]`

To graph the direction field of ODE $x'(t) = f(t, x)$ and several solutions corresponding to the initial conditions $x(t_0) = x_1\,,\ \cdots\,,\ x(t_0) = x_n$, where x_1 through x_n are equally spaced with difference between any two consecutive terms d, do the following.

`graph1=VectorPlot[{1,`$f(t,x)$`}, {t,a,b}, {x,c,d}, VectorScaling->Automatic]`

`Table[NDSolve[{x'[t]==`$f(t,x)$`, x[`t_0`]==x0}, x[t], {t,a,b}], {x0,`x_1`,  `x_n`,  `d`}]`

`(The `x` in function `$f(t,x)$` in NDSolve is typed in as x[t].  The initial values x0 are from `x_1` to `x_n` in steps of `d`.)`

`graph2=Plot[Evaluate[x[t] /.  %], {t,a,b}, PlotRange->{{a,b},{c,d}}]`

`Show[{graph2, graph1}]`

Below are the exact commands needed for this homework!

In[ ]:=
```
graph1a1 = VectorPlot[{1, -y / x}, {x, -10, 10}, {y, -10, 10}, VectorScaling → Automatic];
Table[DSolve[{y'[x] == -y[x] / x, y[-10] == y0}, y[x], x], {y0, -10, 10, 1}];
graph1a2 = Plot[Evaluate[y[x] /. %], {x, -10, 10}, PlotRange → {{-10, 10}, {-10, 10}}];
Show[graph1a2, graph1a1]
```

The modifier **`VectorScaling->Automatic`** in the **`VectorPlot`** results in the direction field, as it is defined. However, sometimes, due to the size differences of the vectors, it might be hard to see the flow. In that case, dropping the modifier solves this problem by producing vectors of the same length, but not the correct direction field (correct directions but incorrect lengths). If the vectors are only intended to be a visual guide and not the direction field, drop the modifier: **`graph1a1 = VectorPlot[{1, -y/x},{x,-10,10},{y,-10,10}]`**.

In[ ]:=
```
ContourPlot[{y^4 + 2 x^2 y^2 == 1, y^4 + 2 x^2 y^2 == 4, y^4 + 2 x^2 y^2 == 9},
 {x, -2, 3}, {y, -3, 2}]
```

In[ ]:=
```
graph1e1 =
  VectorPlot[{1, -(x y) / (x^2 + y^2)}, {x, -2, 3}, {y, -3, 2}, VectorScaling → Automatic];
Table[NDSolve[{y'[x] == -(x y[x]) / (x^2 + y[x]^2), y[-2] == y0}, y[x], {x, -2, 3}],
  {y0, -3, 2, 0.5}];
graph1e2 = Plot[Evaluate[y[x] /. %], {x, -2, 3}, PlotRange → {{-2, 3}, {-3, 2}}];
Show[graph1e2, graph1e1]
```

Here numerical solution, NDSolve, is used since although *Mathematica* could find the explicit solutions, it also had the error messages that it might not have found all solutions.