Math 3280 HOMEWORK #9 Name:

Due 4/3/2026, 11:30 a.m.

Solve the following problems and staple your solutions to this cover sheet. (Computer outputs must be put in the appropriate place in the solution, not attached as an appendix. You may physically cut and paste the output in the problem or allow appropriate space in the printout to add your hand written work.)

1. Solve the following system given in polar coordinates. Show that $r = 1$ is a limit cycle by verifying that it is a closed solution curve and drawing several trajectories. Notes: Use ParametricPlot (or PolarPlot, if solution is re-written in the form $r = f(\theta)$) command to draw the graph of equations in polar coordinates. See below, your first homework, or your book for Mathematica commands.

$$\begin{aligned}\frac{dr}{dt} &= r(1-r^2)\\ \frac{d\theta}{dt} &= 1\end{aligned}$$

2. Find and determine the stability of the limit cycles of the following system given in polar coordinates. Draw the phase portrait, without the vector field, by hand.

$$\begin{aligned}\frac{dr}{dt} &= r(r-2)(r-3)(5-r)^2\\ \frac{d\theta}{dt} &= -1\end{aligned}$$

3. Chap 4, Exer 1. Hints: Convert to polar form. See your class notes. You can draw a limit cycle by "squeezing" it between trajectories. The trajectories can also be used to establish stability.

4. Chap 4, Exer 2. Hints: Use Mathematica to show that the only solution of $x - x^3\cos^3(\pi x) = 0$ between -1 and 1 is $x = 0$. You can draw a limit cycle by "squeezing" it between trajectories. To draw a square by Mathematica see back of this page.

5. Chap 4, Exer 3. Hints: Use Mathematica to show that there are no other critical points except (0, 0). Convert to polar form. Show and use $\frac{1}{2} \le \sin^4\theta + \cos^4\theta \le 1$. See your class notes.

6. Chap 4, Exer 7(b). Hints: A limit cycle must contain a critical point. A limit cycle can not contain just one saddle point.

7. Chap 4, Exer 8(b).

8. Show that the system

$$\begin{aligned}\frac{dx}{dt} &= y + x^3\\ \frac{dy}{dt} &= x + y + x^3\end{aligned}$$

does not have a limit cycle in **two different ways**.

9. Free Points.

10. Free Points.

Mathematica Commands

Consider the autonomous planar system $x'(t) = ax + by$ and $y'(t) = cx + dy$.

Graph the direction field as follows.

```
VectorPlot[{a x + b y, c x + d y}, {x, xmin, xmax}, {y, ymin, ymax},
                                                    VectorScaling->Automatic]
```

(`xmin, xmax, ymin,` and `ymax` are the lower and uper range of values of x and y, respectively.)

You can make all vectors the same length, if desired, by dropping the command `VectorScaling->Automatic`.

```
graph1=VectorPlot[{a x + b y, c x + d y}, {x, xmin, xmax},{y, ymin, ymax}]
```

To find a solution as a function of the initial value $x(0) = x_0$ and $y(0) = y_0$ do the following.

```
gensol[x0_, y0_]:=NDSolve[{x'[t]==a x[t]+b y[t], y'[t]==c x[t]+d y[t],
                          x[0]==x0, y[0]==y0}, {x[t], y[t]}, {t, tmin, tmax}]
```

(`tmin` and `tmax` are the lower and uper range of values of t. Often, it is helpful to use a negative value for `tmin`.)

Then you can find several solutions for the different initial values (x_0, y_0): `(x1, y1); (x2, y2);` ...; `(xn, yn)`.

```
sol[1]=gensol[x1, y1]; sol[2]=gensol[x2, y2]; ··· ; sol[n]=gensol[xn, yn];
```

These solutions can be graphed on the same coordinate system using `ParametricPlot` and `Table` commands.

```
graph2=ParametricPlot[Evaluate[Table[{x[t],y[t]}/.sol[i], {i,n}]],
                      {t, tmin, tmax},PlotRange->{{xmin, xmax}, {ymin, ymax}}]
```

(n is the number of solutions and `{i, n}` steps `i` from 1 to `n` in steps of one. `{x[t], y[t]}/.sol[i]` replaces $(x(t), y(t))$ with the `i`th solution, `sol[i]`.

Now, to display the vector field and the trajectories on the same coordinate system do the following.

```
Show[{graph1, graph2}] .
```