Definitions

2.5.1 Three points A, B, and C are **collinear** if there exists one line ℓ such that all three points A, B, and C all lie on ℓ. The points are **noncollinear** if there is no such line ℓ.

2.6.1 Two lines ℓ and m are said to be **parallel** if there is no point P such that P lies on both ℓ and m. The notation for parallelism is $\ell \parallel m$. A line is not parallel to itself.

4.3.1 Let A and B be sets. A **function from A to B** is a rule f that assigns to each element x of A a unique element $f(x)$ in B. The set A is called the **domain** of the function and the set B is called the **range** of the function.

4.2.6 Let A be a set of real numbers. A number b is called an **upper bound** for A if $x \leq b$ for every $x \in A$. The number b_0 is called the **least upper bound** for A if b_0 is an upper bound for A and $b_0 \leq b$ for every b that is an upper bound for A.

5.4.3 Define the **segment** $\overline{AB}$ by
$$\overline{AB} = \{A, B\} \cup \{P \mid A * P * B\}$$

and the **ray** $\overrightarrow{AB}$ by
$$\overrightarrow{AB} = \overline{AB} \cup \{P \mid A * B * P\}$$

5.4.4 The **length** of segment $\overline{AB}$ is AB, the distance from A to B. Two segments $\overline{AB}$ and $\overline{CD}$ are said to be **congruent**, written $\overline{AB} \cong \overline{CD}$, if the segments have the same length.

5.4.5 The points A and B are the **endpoints** of the segment $\overline{AB}$; all other points of $\overline{AB}$ are **interior points**. The point A is the **endpoint** of the ray $\overrightarrow{AB}$.

5.4.8 A **metric** is a function $D: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ such that
1) $D(P, Q) = D(Q, P)$ for every P and Q.
2) $D(P, Q) \geq 0$ for every P and Q, and
3) $D(P, Q) = 0$ if and only if $P = Q$.

5.4.11 Let ℓ be a line. A one-to-one correspondence $f: \ell \to \mathbb{R}$ such that $PQ = |f(P) - f(Q)|$ for every P and Q on ℓ is called a **coordinate function** for the line ℓ and the number $f(P)$ is called the **coordinate** of the point P.

5.5.8 Three points A, B, and C are **collinear** if there exists one line ℓ such that A, B, & C all lie on ℓ. The points are **noncollinear** otherwise.

5.5.9 Let A, B, and C be three noncollinear points. The **triangle** $\triangle ABC$ consists of the union of three segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$; that is

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

The points A, B & C are called the **vertices** of the triangle and the segment $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called the **sides** of the triangle.

5.6.1 Ray $\overrightarrow{AD}$ is **between** rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if D is in the interior of $\angle BAC$.

5.6.3 Two angles $\angle BAC$ and $\angle EDF$ are said to be **congruent**, written $\angle BAC \cong \angle EDF$, if $\mu(\angle BAC) = \mu(\angle EDF)$.

5.6.4 Angle $\angle BAC$ is a **right angle** if $\mu(\angle BAC) = 90°$, $\angle BAC$ is an **acute angle** if $\mu(\angle BAC) < 90°$, and $\angle BAC$ is an **obtuse angle** if $\mu(\angle BAC) > 90°$

5.7.4 Let A and B be two distinct points. The point M is a **midpoint** of $\overline{AB}$ if M is between A and B and $AM = MB$.

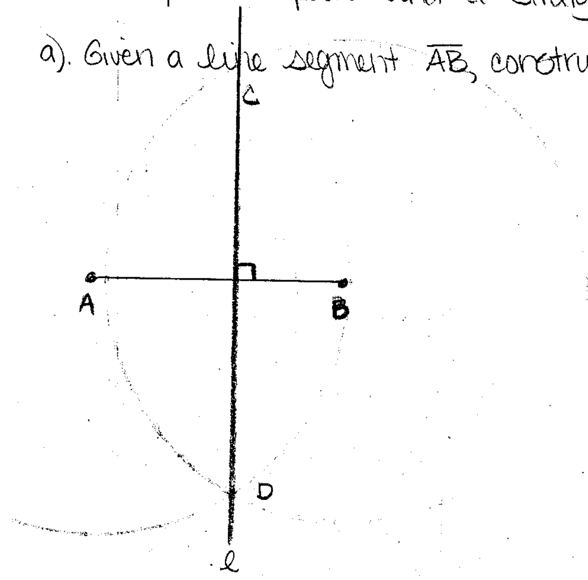
5.8.1 Two triangles are **congruent** if there is a Correspondence between the vertices of the first triangle and the Vertices of the second triangle such that Corresponding angles are congruent and Corresponding sides are Congruent. Thus the assertion that two triangles are congruent is really the assertion that there are six Congruences, three angle congruences and three segment congruences.

5.8.4 A triangle is called **isosceles** if it has a pair of Congruent sides.

HW #1

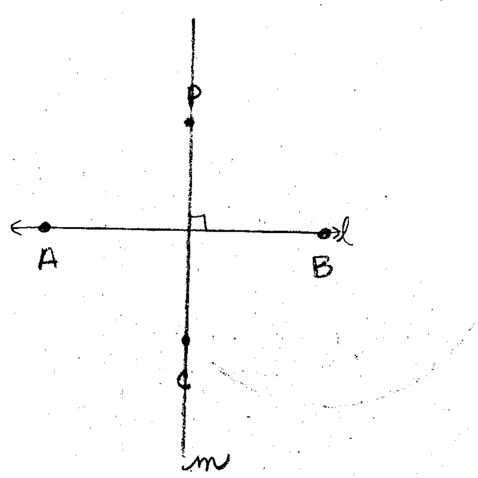
5. Ch. 1.6 Explain how to complete the following constructions using only a compass and a straightedge.

a). Given a line segment $\overline{AB}$, construct the perpendicular bisector of $\overline{AB}$



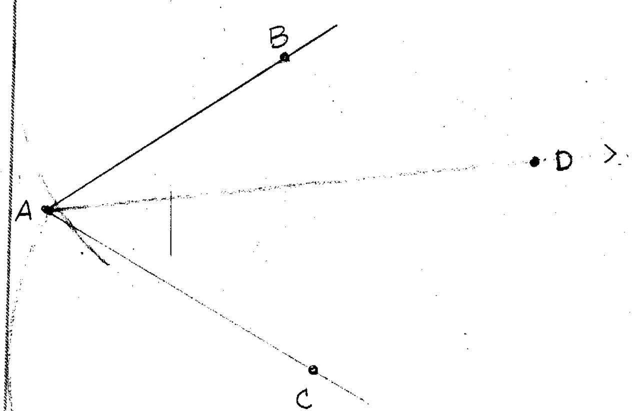
Put point of compass on point A. Measure to point B, by placing pencil part of compass on B. Draw a circle w/center A & radius $\overline{AB}$. Keep distance the same on compass, place the point of the compass on B, draw a circle w/ center B & radius $\overline{AB}$, Let C & D be the points where the two circles intersect. Place a straightedge on C & D and draw a line, ℓ. ℓ is now a perpendicular bisector of $\overline{AB}$. :)

b). Given a line ℓ and a point P not on ℓ, construct a line through P that is perpendicular to ℓ.



Let the points A & B lie on ℓ. Put point of compass on A & measure to point P with pencil end of compass. Draw a circle w/center A radius from A to P. Put point of compass on B and measure to point P. Draw a circle w/center B radius from B to P. Let C be the point of intersection of the two circles on the opposite side of ℓ that P is on. Place a straightedge on P & C and draw a line, m. m is now perpendicular to ℓ and goes through the point P. :)

c). Given an angle $\angle BAC$, construct the angle bisector



Put point of compass on B. Measure to point A. Draw a circle w/ center B radius $\overline{AB}$. Put point of compass on point C. Measure to A. Draw a circle with center C, radius $\overline{AC}$. Let D be the point where the two circles intersect. Draw a line from A to D and call it $\overrightarrow{AD}$. $\overrightarrow{AD}$ is the angle bisector of the angle $\angle BAC$. :)

HW #1

6. Ch.1.7. Can you prove the following assertions using only Euclid's postulates & common notions.

a). Every line has at least two points lying on it.

Postulate I states: To draw a straight line from any point to any point.

Meaning that every line has at least two points.

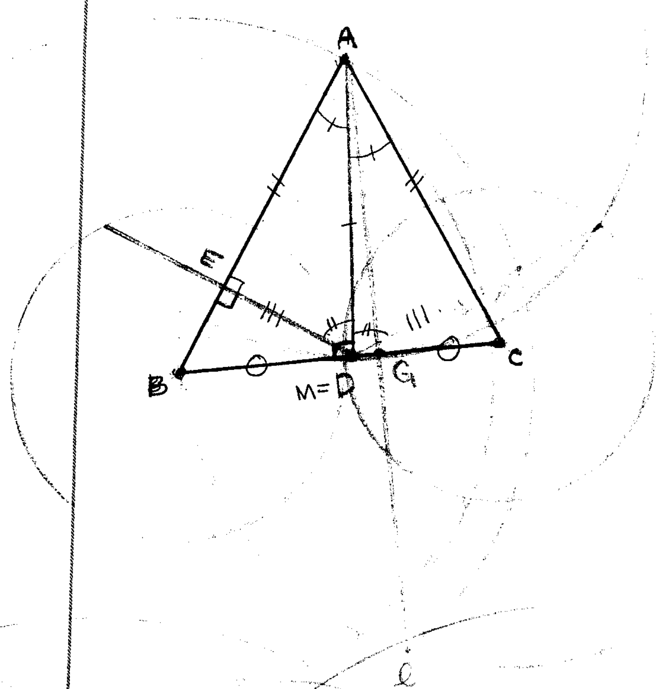
→ b). For every line there is at least one point that does not lie on the line.

→ c). For every pair of points $A \neq B$, there is only one line that passes through A & B.

HW #1

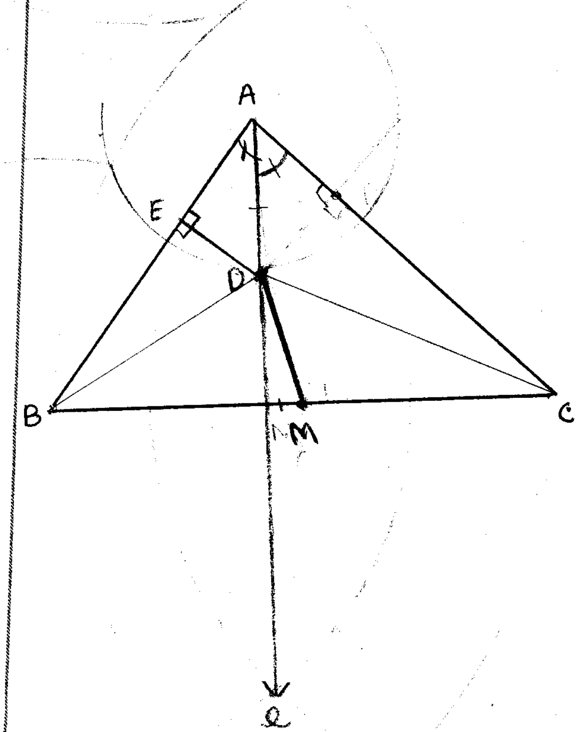
7. Ch. 1.10. Find the flaw in the proof

pg 15 What is "Hypotenuse-Leg Thm"? Claims $\triangle BDE \cong \triangle CDF$ and therefore $\overline{BE} \cong \overline{CF}$. Hence $\overline{AB} \cong \overline{AC}$ by addition. I do know that $\triangle BDE \cong \triangle CDF$ because of SAS. so $\overline{AB} \cong \overline{AC}$.



CASE I OKAY

CASE II



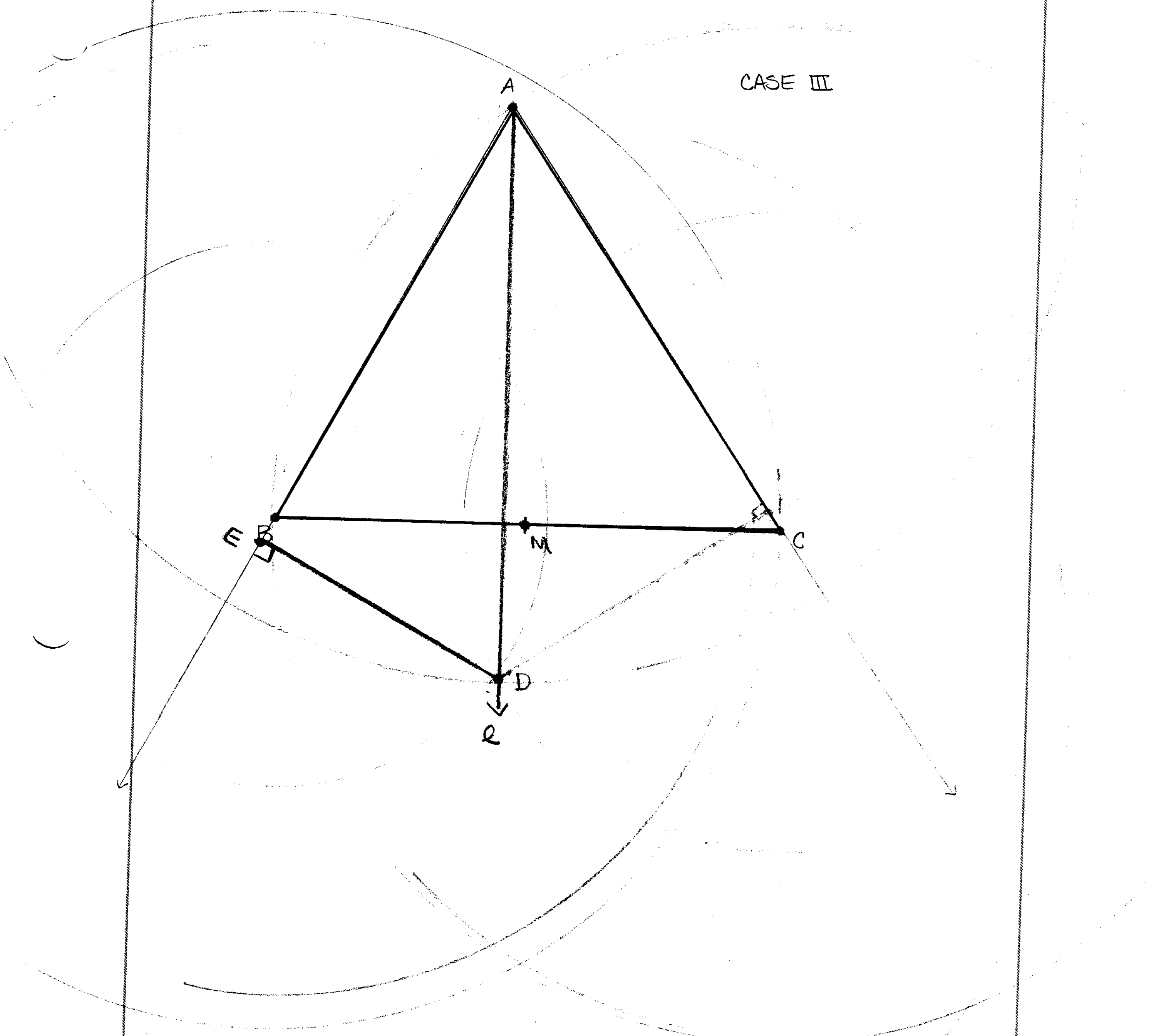
Is M supposed to have a ⊥ ? from D to M? Yes, because every 2 points makes a line? Why not line from M to F or from M to E?

We do not know that $\overline{AE} \cong \overline{AF}$
We have AAS.

How do we know to draw a line from B to D and from C to D?

HW #1

7 continued



Fails because E & F cannot be both outside triangle ABC.

Assignment 1 #4 a)

$a = u^2 - v^2;\ b = 2uv;\ c = u^2 + v^2$

$a^2 + b^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 =$

$= u^4 + 2u^2v^2 + v^4 = (u^2 + v^2) = c^2$ So $a^2 + b^2 = c^2$

b)

Let $u = 2k + 1;\ v = 2j + 1$

$a = u^2 - v^2 = (2k+1)^2 - (2j+1)^2 = 4k^2 + 4k + 1 - 4j^2 - 4j - 1 = 4(k^2 - j^2 + k - j)$

$b = 2uv \Rightarrow$ b is always even

$c = u^2 + v^2 = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 2(2(k^2 + j^2 + k + j) + 1)$

So a 2 can be factored out of *a, b,* and *c* therefore they are all even when *u* and *v* are odd.

c)

if $u = 2k;\ v = 2j + 1$

$a = u^2 - v^2 = (2k)^2 - (2j+1)^2 = 4k^2 - 4j^2 - 4j - 1$

$b = 2uv = 4k(2j+1) = 8jk + 4k$

$c = u^2 + v^2 = (2k)^2 + (2j+1)^2 = 4k^2 + 4j^2 + 4j + 1 = 4k^2 + 4j^2 + 4j + 1$

If *u* and *v* are relatively prime the only factors that divide b are 2, divisors of *u* and divisors of *v;* so say p divides *u* and q divides *v* we know that p is relatively prime to *v* and q is relatively prime to *v* thus $c \bmod(p) \equiv v^2 \neq 0$ and $c \bmod(q) \equiv u^2 \neq 0$ and since *a* and *c* are both odd 2 does not divide either of them. So we have shown that any number that divides *b* will not divide c thus if *u* and *v* are relatively prime GCD(*a,b,c*)=1 and from (a) we know *(a,b,c)* is a Pythagorean triple, therefore *(a,b,c)* is a primitive Pythagorean triple.

Homework #1

(11.) 2.5 Interpret *point* to mean one of the four vertices of a square, *line* to one of the sides of the square, and *lie on* to mean that one vertex is an endpoint of the side. Which incidence axioms hold in this interpretation? Which parallel postulates hold?

Point: any symbols of the vertices A, B, C or D of the square

line: (sides) {A, B}, {B, C}, {C, D}, {D, A}

lie on: one vertex is endpoint of the side.

- axiom 1 doesn't hold; no line contains A & C.
- axiom 2 holds; (every line contains @ least two points.)
- axiom 3 holds; (there exists three point not all on l.)

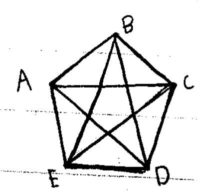
• The Euclidean Parallel Postulate holds. ■

(12.) 2.6 Draw a schematic diagram of five-point geometry. (See example 2.6.3)

Points: symbols A, B, C, D, & E.

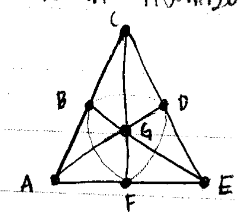
Lines: {A, B}, {A, C}, {A, D}, {A, E}, {B, C}, {B, D}, {B, E}, {C, D}, {C, E}, {D, E}.

Lie on: element of.



(13) $\boxed{2.10}$ Consider a finite model for incidence geometry that satisfies the following additional axiom. Every line has exactly three points lying on it. What is the min. # of points in such a geometry? Explain.

- There needs to be @ least three sides of three points which makes six points. In order to create lines between each set of points, we need to add one more point that they can all pass through to create incidence geometry. The minimum number of points is $\boxed{7}$.



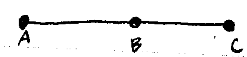
(15) $\boxed{2.12}$ Find interpretations of words point, line, + lie on that satisfy the following conditions:

a) Incidence axioms 1 + 2 hold, but axiom 3 doesn't.

points: symbols A, B, + C.

line: set of all three points {A,B,C}

lie on: the three points elements of line.

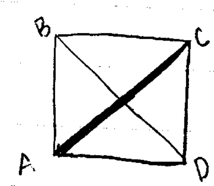


b) Incidence axioms 2 + 3 hold, but axiom 1 doesn't.

points: symbols A, B, C, + D.

lines: {A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {A,C}

lie on: the points are elements of the lines

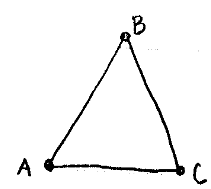


c) Incidence axioms 1 + 3 hold, but axiom 2 doesn't.

points: symbols A, B, + C

lines: {A,B}, {A,C}, {B,C}, {A}

lie on: the points are incidence w/ the lines



2.12 a) Points: symbols A B C
line: set of all points
a point lies on a line when it is part of set comprising line

b) point: 4 vertices of a square
line: sides of square
lie on: vertex is an endpoint of side

c) points: toppings
lines: pizzas
lie on: a topping is on a pizza

Consider 4 pizzas
Pizza 1: has pepperoni & olives
Pizza 2: has pepperoni & mushrooms
Pizza 3: has mushrooms & olives
Pizza 4: has pepperoni

Homework II

1. These are the negations

   a) For every model of Incidence geometry the Euclidean Parallel Postulate does not hold.

   b) There exists a model of Incidence geometry in which there are not exactly 7 points.

   c) There exists a triangle whose angle sum is not 180°.

   d) It is not hot or humid outside.

   e) My favorite color is not red and not green.

   f) It is possible that the sun shines, and we don't go hiking.

   g) There is a geometry student who does not know how to write proofs.

2. not (S and T) = (not S) or (not T)

S	T	S and T	not (S and T)	not S	not T	(not S) or (not T)
T	T	T	F	F	F	F
T	F	F	T	T	T	T
F	T	F	T	F	F	T
F	F	F	T	T	T	T

Hence, not (S and T) = (not S) or (not T)

not (S or T) = (not S) and (not T)

S	T	S or T	not (S or T)	not S	not T	(not S) and (not T)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Hence, not (S or T) = (not S) and (not T)

Homework 3

2.4.11 Prove $\sqrt{3}$ is irrational

If x is a real number such that $x^2=3$, then x is irrational.

Let x be a real number such that $x^2=3$. (hypothesis). Suppose x is rational (RAA). Then x can be written as $\frac{p}{q}$, where p and q are both integers, and $\frac{p}{q}$ is the lowest terms. Then p and q have no common factors. Then p and q have no common factors. Now we have $\frac{p^2}{q^2}=3$ which implies $p^2=3q^2$. So, by def, p^2 is divisible by 3, which implies p is also divisible by 3. Let $p=3s$, then $(3s)^2=3q^2 = 9s^2=3q^2$. Then $q^2=3s^2$, so q^2 is divisible by 3, and q is also divisible by 3. Therefore p and q have a common factor. This is a contradiction. We must reject that x is rational and conclude x is irrational. Ü

Homework II

9. (4.3 in book)

Convert $2.3\overline{571}$ to a fraction.

Let $x = 2.3\overline{571}$

Then $10x = 23.\overline{571}$

and $10^4 x = 23571.\overline{571}$.

$10^4 x - 10x = 23571.\overline{571} - 23.\overline{571} = 23548$

$9990x = 23548$

$x = \frac{23548}{9990} = \frac{11774}{4995}$

Homework III

#4. (5.2 in book)

$$\rho((x_1,y_1),(x_2,y_2)) = |x_2-x_1| + |y_2-y_1|$$

1. $\rho(P,Q) \overset{?}{=} \rho(Q,P)$ for every P and Q
   Let $P=(x_1,y_1)$ and $Q=(x_2,y_2)$.
   Then $\rho(P,Q) = |x_2-x_1| + |y_2-y_1|$
   and $\rho(Q,P) = |x_1-x_2| + |y_1-y_2|$.
   By property of abs. values
   $\rho(P,Q) = |x_2-x_1| + |y_2-y_1| = |x_1-x_2| + |y_1-y_2| = \rho(Q,P)$

2. $\rho(P,Q) \geq 0$ for every P and Q
   $\rho(P,Q) = |x_2-x_1| + |y_2-y_1|$
   Because abs. values are always positive or 0,
   we are adding 2 positive numbers. Hence
   $\rho(P,Q) \geq 0$ for every P and Q

3. $\rho(P,Q) = 0$ iff $P=Q$.
   If $\rho(P,Q)=0$, then $|x_2-x_1| + |y_2-y_1| = 0$.
   Then $|x_2-x_1| = 0$ and $|y_2-y_1|=0$, which
   implies $x_2-x_1=0 \Rightarrow x_1=x_2$. Similarly,
   $y_2=y_1$. Hence $P=Q$.
   If $P=Q$, then $x_1=x_2$, $y_2=y_1$, and $\rho(P,Q) = |x_1-x_1| + |y_1-y_1|$.
   $|x_1-x_1|=0$ and $|y_1-y_1|=0$.
   We have $\rho(P,Q) = 0+0=0$. Hence
   $\rho(P,Q)=0$.

Assignment 2 #4 3.12

Prove Theorem: There exist three distinct lines such that no point lies on all three of the lines.

Proof. We will show that we have three non-collinear points and that each pair of points defines a unique line then there are three distinct lines. Our third axiom says that we have three non-collinear points call them a, b, and c. Our first axiom says that for every pair of points there exists a unique line through those points and since a, b, and c are non-collinear we have three distinct lines denote those lines by $\{a,b\};\{b,c\};\{a,c\}$. Point a is not on line $\{b,c\}$, point b is not on line $\{a,c\}$, and point c is not on line $\{a,b\}$. If there exists another point p such that p is on line $\{a,b\}$ then point p cannot lie on line $\{a,c\}$ or $\{b,c\}$ by theorem 3.6.1 a is the unique point that $\{a,c\}$ and $\{a,b\}$ share similarly for b on $\{a,b\}$ and $\{b,c\}$. So there are three distinct lines that do not share a common point.□

Assignment 2 #7 <u>Theorem 2</u>

If A scorples B and C is distinct from A, then A scorples C or C scorples B.

Proof. Given that A scorples B and that C is distinct from A we can say by SF1* that A scorples C or C scorples A. If C scorples A and we know A scorples B then by SF3 C scorples B. So A scorples C or C scorples B. □

Assignment 2 #8 <u>Theorem 3</u>

There is at least one flug that scorples every other flug.

Proof. By SF4 we know that there are four distinct flugs. By SF1* we know that one flug will scorple another flug or be scorpled by that other flug. So without losing generality we will name our flugs A, B, C, and D such that A scorples B and C scorples D. Now by SF1* we know B scorples C or C scorples B but not both. If B scorples C then using SF3 and that A scorples B we have that A scorples C. This by the same reasons gives A scorples D, so A scorples every other flug. If C scorples B then by SF1* we know A scorples C or C scorples A. If A scorples C then by SF3 A scorples D and so A scorples every other flug. If C scorples A then we have C scorples B and C scorples D so C scorples every other flug. So in each case we have a flug that scorples every other flug. □

3.1

a) In every model for incidence Geometry the EPP does not hold.
b) There exists a model for incidence Geometry that does not have exactly 7 points.
c) There exists a triangle that has an angle sum that does not equal 180 degrees.
d) It is not hot or not humid outside.
e) My favorite color is not red and not green.
f) The sun shines and we don't go hiking.
g) There exists a geometry student who doesn't know how to write proofs.

3.5

a) H: If I pass geometry
C: Then I can take topology.

b) H: If it rains
C: I get wet.

c: H: If a number is divisible by 4
C: It must be even.

Homework # 4

1. Let l = line AB, let f be a coordinate function f: $l \rightarrow \Re$ with f(A) = 0 and f(B) > 0. $\exists x$ such that $0 < x < f(B)$ by the Density Postulate. Let $C = f^{-1}(x)$.
$[0 = f(A)] < [f(C) = x] < f(B)$. By the Betweenness Theorem for points, C is between A and B. Now, $\exists y \in \Re$ such that $f(B) < y$. Let $D = f^{-1}(y)$. So $[0 = f(A)] < f(B) < f(D)$. By the Betweenness theorem, B is between A and D. Now, $\exists z \in \Re$ such that $z < 0 < f(B)$. Let $E = f^{-1}(z)$. So $[z = f(E)] < [f(A) = 0] < f(B)$.

$\therefore$ A * C * B, A * B * D, and E * A * B.

Assignment 3 #3. 4.15

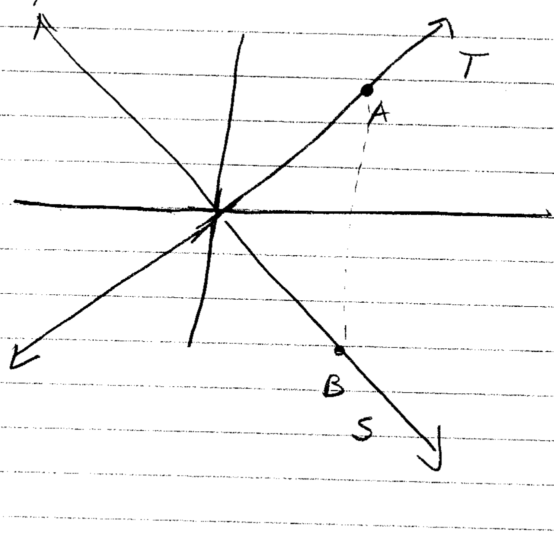
Let $\varepsilon > 0$ and $A = \{n\varepsilon \mid n \in \aleph\}$. We will show that for any M there is a natural number n such that $n\varepsilon > M$. To do this we establish the upper bound of A is empty therefore M is not in the upperbound of A giving there is an element of A greater than M. Suppose the upper bound of A is nonempty. By the least upper bound postulate there exists a number b which is the least upper bound of A. Since b is the least upper bound any element less than b will be less than an element of A, so there exists an $a \in A$ such that $a > b - \varepsilon$ which gives $a + \varepsilon > b$. But if $x \in A$ then $x = m\varepsilon$ for some $m \in \aleph$ and if $m \in \aleph$ then $m + 1 \in \aleph$ by the induced definition of the natural numbers, this means $(m+1)\varepsilon \in A$ so $x + \varepsilon \in A$. This gives if $a \in A$ then $a + \varepsilon \in A$ and $a + \varepsilon > b$ which contradicts that b was the least upper bound. So A has no upper bound and therefore there exists n such that $n\varepsilon > M$.

5.9 Proof. Let T & S be convex sets. There are two possibilities – that $T \cap S = \emptyset$ or $T \cap S \neq \emptyset$.
Case 1: Assume $T \cap S = \emptyset$. Since there exists no pair of points in the empty set, then vacuously the empty set is convex by definition.

Case 2: Assume $T \cap S \neq \emptyset$. Let A be a point such that $A \in S$ and $A \in T$. Let B be a point such that $B \in S$ and $B \in T$. Since T & S are convex, the entire segment $\overline{AB} \subset T$ & S. Thus, $\overline{AB} \subset T \cap S$ and so $T \cap S$ is a convex set.

- Example that $T \cup S$ need not be convex

On the Cartesian plane, let T equal the set of points that lie on the line $y = x$. Let S equal the set of points that lie on the line $y = -x$. Let $A \in T$ such that $A \neq (0,0)$ and $B \in S$ such that $B \neq (0,0)$. The segment $\overline{AB} \notin T \cup S$.



12/12

5.21 *Proof*: Let $A*B*C$, $D*E*F$, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$. We will prove that $\overline{AC} \cong \overline{DF}$.

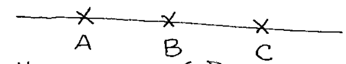
By definition of betweenness of points $AB + BC = AC$ and $DE + EF = DF$. Since $\overline{AB} \cong \overline{DE}$, $AB = DE$ and since $\overline{BC} \cong \overline{EF}$, $BC = EF$. Therefore by substitution $AC = DE + EF = DF$. Therefore $\overline{AC} \cong \overline{DF}$

5.22 *Proof*: Let A, B, C, D, E, F be distinct points such that $A*B*C$, $D*E*F$, $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. We will prove that $\overline{BC} \cong \overline{EF}$. By definition of betweenness $AB + BC = AC$ and $DE + EF = DF$. By definition of congruence $AB = DE$ and $AC = DF$. We know that $BC = AC - AB$ from our definition of betweenness. Similarly we know $EF = DF - DE$. Using substitution we get $EF = AC - AB = BC$. Since $EF = BC$, $\overline{EF} \cong \overline{BC}$. $\square$

HW # V $\quad$ 8/12

6. Problem 5.29 Prove existence & uniqueness of midpoints (Theorem 5.7.5)

Proof Let A & B be two distinct points; Let



$f: \overleftrightarrow{AB} \to \mathbb{R}$ be a coordinate function of the line $\overleftrightarrow{AB}$. Let $x = \frac{f(A)+f(B)}{2}$, Since f is onto, there is a point $M \in \overleftrightarrow{AB}$ such that $f(M) = x$. The number $x = f(M)$ being the average of f(A) & f(B) is between f(A) & f(B). M is between A & B, by betweenness theorem for point.

$$AM = |f(A) - f(M)| = \left| f(A) - \frac{f(A)-f(B)}{2} \right| = \left| \frac{f(A)-f(B)}{2} \right|$$

$$MB = |f(M) - f(B)| = \left| \frac{f(A)-f(B)}{2} - f(A) \right| = \left| \frac{f(A)-f(B)}{2} \right|$$

So $AM = MB$. Hence M is a midpoint of $\overline{AB}$. Suppose N is another midpoint of $\overline{AB}$. Point N is between A & B and $AN = NB$. ~~or f(B) < f(N)~~ The Betweenness Theorem for points implies that $f(A) < f(N) < f(B)$ ~~< f(A)~~

o.k. or $f(B) < f(N) < f(A)$ In either case, $|f(A) - f(N)| = |f(N) - f(B)|$. implies that $f(N) - f(A) = f(B) - f(N)$ which implies $f(A) - f(N) = f(N) - f(B)$. Thus, $f(N) = \frac{f(A)-f(B)}{2}$, by algebra. That is $f(N) = f(M)$. Since f is 1-1, this means $N = M$. There is no other point that doesn't equal M, therefore M is unique.

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Note to Sherri - this goes back in HW #5