Chapter 1 Introduction

Sec 1.1 Background

Def Any equation containing one or more derivatives of an unknown function is called a differential equation.

Examples	ODE/PDE	order	Linear/Nonlinear

1. $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q(t)=E(t)$

   where $Q=Q(t)$

2. $u_{rr}+\dfrac{1}{r}u_r+\dfrac{1}{r^2}u_{\theta\theta}=0$

   where $u=u(r,\theta)$

3. $y'+3y=x+e^{-2x}$

   where $y=y(x)$

4. $\dfrac{d^2\theta}{dt^2}+\dfrac{g}{l}\sin\theta=0$

   where $\theta=\theta(t)$

← 1.1.2

Def 1. Any DE which only involves ordinary derivatives of the unknown function is called an ordinary differential equation.

2. Any DE which involves partial derivatives of the unknown function is called a partial differential equation.

Def The order of a DE is the order of the highest derivative that appears in the equation.

Any n-th order ODE is of the form

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad \text{where } y = y(x)$$

1.1.3 →

Def A function $L = L(y)$ is called linear (linear in y) if $L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$ for all constants c_1 & c_2 and y_1 & y_2 in the domain of L.

Ex Determine if a given function is linear or not.

1. $f(y) = y^2$

1.1.4

2. $g(y) = 5y - 1$

3. $h(y) = 3y$

Def The ODE $F(x, y, y', \dots, y^{(n)}) = 0$ is said to be linear if F is a linear function of the variables $y, y', \dots, y^{(n)}$.

Fact: The general linear ODE of order n is of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Any ODE not of the above form is a nonlinear equation.

It is also possible to have a system of ODE's

$$\begin{cases} \dfrac{dx}{dt} = x(1-x-y) \\ \dfrac{dy}{dt} = y(0.5-0.75x-0.25y) \end{cases}$$

where $x = x(t)$ and $y = y(t)$.

1.1.6

Sec 1.2 Solutions and Initial Value Problems

Def A solution of a DE is a function which satisfies the DE.

A solution may be stated explicity or implicitly.

Ex show that $y = 2e^{-3x} + e^{-2x} + \frac{x}{3} - \frac{1}{9}$ is an explicit solution of $y' + 3y = x + e^{-2x}$.

← 1.2.2

Ex show that $x^2 - 2y^3 = 5$ is an implicit sol of

$$\frac{dy}{dx} = \frac{x}{3y^2}.$$

Note: In the last example, we could solve for y explicitly

and the check it!

<u>Def</u> Initial Value Problem (of order n) - An ODE of order n plus n initial conditions, as shown below.

$$F(x, y, y', \dots, y^{(n)}) = 0$$

$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = y_{n-1}$$

with $x_0, y_0, y_1, \dots, y_{n-1}$ given values

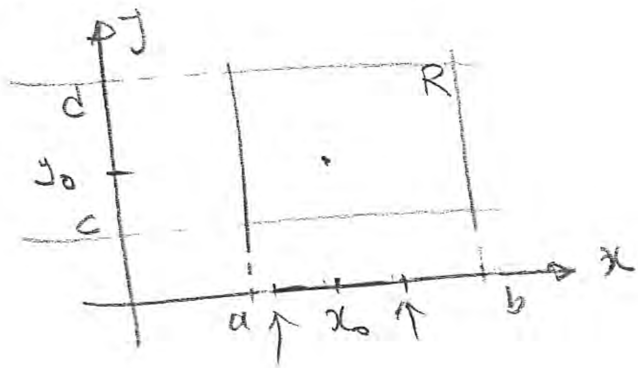
<u>Ex</u> Show that $y = -\frac{1}{2} + e^{x^2}$ is a solution of the IVP $y' - 2xy = x$, $y(0) = \frac{1}{2}$.

← 1.2.4

<u>Existence and Uniqueness Theorem</u> Let function f and $\frac{\partial f}{\partial y}$ (or f_y) be continuous is some rectangle $R = \{(x,y) : a<x<b,\ c<y<d\}$ containing the point (x_0, y_0). Then the IVP

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y = \phi(x)$ in some interval $(x_0 - h,\ x_0 + h)$, where h is a positive number.



Continuity & Partial Derivative of Functions of Two Variables

1.2.5 →

← 1.2.6

Ex Show that $y' + \frac{1}{x} y = \sin x$, $y(\frac{\pi}{2}) = 1$, has a unique solution.

Ex 1. Show that $y = \pm\left(\frac{4}{3} x^2\right)^{3/2}$ and $y = 0$ are solutions of $y' = 4xy^{1/3}$, $y(0) = 0$.

1.2.7 →

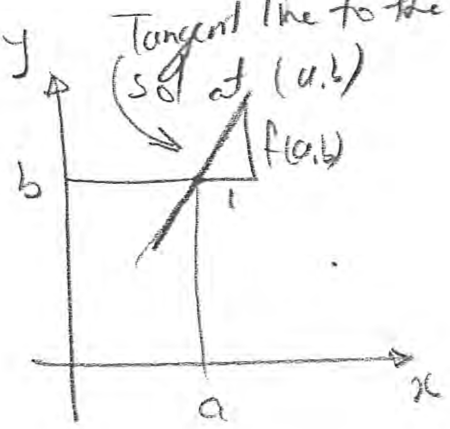
2. Why part 1 does not contradict the Existence & Uniqueness Theorem?

1.2.8

Sec 1.3 Direction (Slope) Fields

Without solving an ODE we can discover some info about the nature of its solution.

Consider $\frac{dy}{dx} = f(x, y)$.



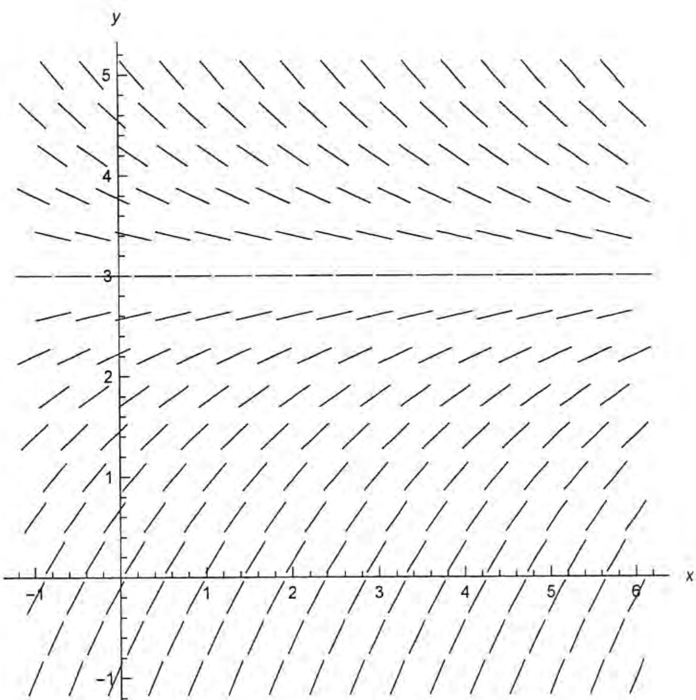
The slope of the solution (slope of the tangent line to the sol) at (a, b) is

$$\text{slope}\Big|_{(a,b)} = \frac{dy}{dx}\Big|_{(a,b)} = f(a,b).$$

Thus, at (a, b) we can draw a short line segment tangent to the sol at (a, b).

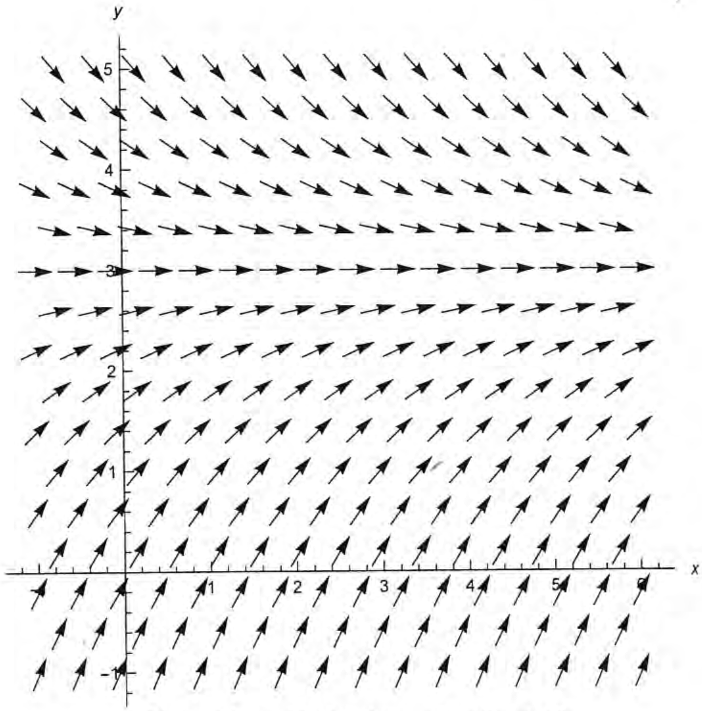
← 1.3.2

The collection of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a direction (slope) field.



Slope field for y' = (3-y)/2

1.3.3 →



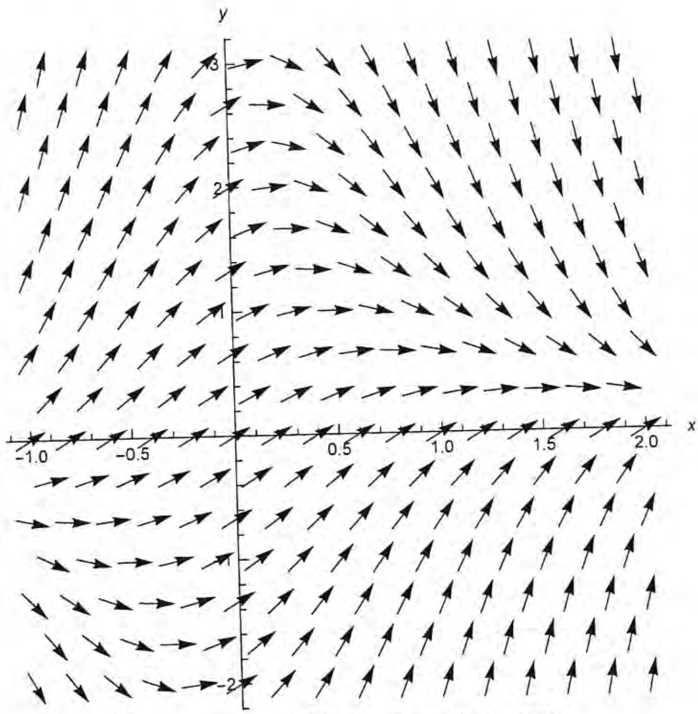
Direction field for y'=(3-y)/2

Solution Curves

Flows

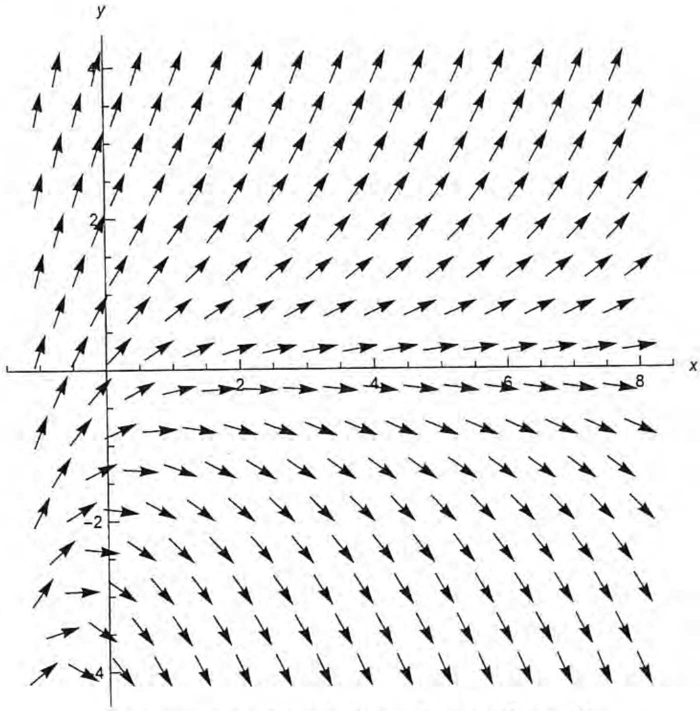
Trajectories

← 1.3.4



Direction field for y'=1-2xy

1.3.5 →



Direction field for $y' = (2e^{-x} + y)/2$

← 1.3.6

The Method of Isoclines

An isocline for $y' = f(x,y)$ is a set of points in the xy-plane where all solutions have the same slope $\frac{dy}{dx}$. That is, it is a level curve for the function $f(x,y)$; $\frac{dy}{dx} = f(x,y) = c$.

Ex Use the method of isoclines to draw a direction field for $\frac{dy}{dx} = 2x^2 - y$ and sketch several of its solution curves.

1.3.7

← 1.3.8

Chapter 2 First Order Differential Equations

Sec 2.1 Introduction: Motion of a Falling Body

Read it ! See the definition of the General Solution.

Sec 2.2 Separable Equations

In this Chapter we will consider ODE's of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

A subclass of this equation is

$$\frac{dy}{dx} = f(x,y) \qquad \left(f(x,y) = -\frac{M(x,y)}{N(x,y)} \right)$$

And a special case of this equation is

← 2.2.2

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \qquad \left(f(x,y) = \frac{g(x)}{h(y)} \right)$$

In differential form, this equation is

$$h(y)\,dy = g(x)\,dx$$

Def A 1st order ODE $\frac{dy}{dx} = f(x,y)$ is called separable if $f(x,y) = \frac{g(x)}{h(y)}$. That is, x's and y's in the ODE can be separated.

Ex Find the general solution of the 1st order separable ODE $e^{y}\frac{dy}{dx} - x - x^{3} = 0$

To Solve a 1st Order Separable ODE

1. Write it in the form $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$.
2. Multiply both sides by dx & $h(y)$ (separate x's and y's) to get $\quad h(y)\,dy = g(x)\,dx$
3. Integrate both sides $\int h(y)\,dy = \int g(x)\,dx$

$$H(y) = G(x) + C$$

4. If given an initial condition, use it to find C.
5. If cannot integrate in part 3 and the initial condition is $y(x_0) = y_0$, then use definite integrals as follows

$$\int_{y_0}^{y} h(t)\,dt = \int_{x_0}^{x} g(t)\,dt + C$$

This will result in $C = 0$ since plugging in

2.24

$y(x_0)=y_0$ we get $\int_{y_0}^{y_0} h(t)\,dt = \int_{x_0}^{x_0} g(t)\,dt + C \Rightarrow C=0$

For a example of this case, see problem #27 in your homework.

<u>Ex</u> Solve $\frac{dy}{dx} = 1+y^2$. $y(0)=0$

2.2.5 →

Ex Find the general solution of $y' = \dfrac{x^2}{1-y^2}$.

Ex Solve $(x-2)\dfrac{dy}{dx} + \dfrac{1}{x+3}y = 0$. Assume $x > 2$

2.2.6

Notice that $y=0$ is also a sol of this ODE.

Justification of the Method

(Why we could multiply by dx?)

Reconsider $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

Multiply both sides by $h(y)$ $\quad h(y)\frac{dy}{dx} = g(x)$

Let $H(y)$ be an antiderivative of $h(y)$:

$$\frac{d}{dy}H(y) = h(y)$$

Let $G(x)$ be an antiderivative of $g(x)$:

$$\frac{d}{dx}G(x) = g(x)$$

By the Chain rule,

$$\frac{d}{dx}H(y) = \frac{d}{dy}H(y)\,\frac{dy}{dx} = h(y)\,\frac{dy}{dx}$$

$\leftarrow$ 2.2.8

So, $\dfrac{d}{dx} H(y) = \dfrac{d}{dx} G(x)$

From Calculus I, $H(y) = G(x) + C$

or, $\int h(y)\,dy = \int g(x)\,dx + C$

Sec 2.3 Linear Equations

Def A first order linear ODE is an equation that is of the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=b(x)$$

Note: It is of the form $M(x,y)+N(x,y)\frac{dy}{dx}=0$ with

$M(x,y)=a_0(x)y-b(x)$ and $N(x,y)=a_1(x)$.

The eq $\quad (x^2+1)\frac{dy}{dx}+xy=\sin x$

is

while the eq $\quad (x^2+1)\frac{dy}{dx}+xy^2=x$

is

2.3.2

<u>Ex</u> Find the general solution of $x^2y' + 2xy = \sin x$.

Notice that $\frac{d}{dx}(x^2y) = 2xy + x^2y'$

$$x^2y' + 2xy = \sin x$$

$$\frac{d}{dx}(x^2y) = \sin x$$

Notes: 1. If we write the ODE as $\frac{d}{dx}(*) = **$, then it is easy to solve.

2. If the eq was state as $y' + \frac{2}{x}y = \frac{\sin x}{x^2}$, we would multiply both sides by x^2.

But, how do we come up with this multiplication factor?

Consider
$$a_1(x)\frac{dy}{dx}+a_0(x)y=b(x)$$

Divide both sides by $a_1(x)$ to get

$$\frac{dy}{dx}+P(x)y=Q(x), \quad P(x)=\frac{a_0(x)}{a_1(x)}, \quad Q(x)=\frac{b(x)}{a_1(x)}$$

Find a func $\mu(x)$ so that after multiplying both sides by $\mu(x)$

$$\mu(x)\frac{dy}{dx}+\mu(x)P(x)y=\mu(x)Q(x)$$

the left side can be written as

$$\mu(x)\frac{dy}{dx}+\mu(x)P(x)y=\frac{d}{dx}(\mu(x)y)$$

This happens if

$$\mu(x)\frac{dy}{dx}+\mu(x)P(x)y=\frac{d\mu}{dx}y+\mu(x)\frac{dy}{dx}$$

$$\mu(x)P(x)y=\frac{d\mu}{dx}y$$

$$\frac{d\mu}{dx}=\mu(x)P(x)$$

2.3.4

This is a separable ODE that we can solve

$$\frac{d\mu}{\mu} = P(x)\,dx$$

$$\int \frac{d\mu}{\mu} = \int P(x)\,dx$$

$$\ln|\mu| = \int P(x)\,dx + C$$

$$|\mu| = e^{\int P(x)dx + C} = e^{C} e^{\int P(x)dx}$$

$$\mu = \pm e^{C} e^{\int P(x)dx} = A e^{\int P(x)dx}$$

$$\mu = e^{\int P(x)dx}, \quad \text{using } A = 1$$

With this integrating factor μ, we get

$$\frac{d}{dx}\left(\mu(x) y\right) = \mu(x) Q(x)$$

$$\mu(x) y = \int \mu(x) Q(x)\,dx + C$$

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) Q(x)\,dx + C\right]$$

If we cannot do the integral and the initial condition is $y(x_0)=y_0$, we will use a definite integral

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

$$\mu(x)y = \int_{x_0}^{x} \mu(t)Q(t)\,dt + C$$

Plugging in $y(x_0)=y_0$ we have

$$\mu(x_0)y_0 = \int_{x_0}^{x_0} \mu(t)Q(t)\,dt + C$$

$$C = \mu(x_0)y_0$$

$$\mu(x)y = \int_{x_0}^{x} \mu(t)Q(t)\,dt + \mu(x_0)y_0$$

$$y = \frac{1}{\mu(x)}\left[\int_{x_0}^{x} \mu(t)Q(t)\,dt + \mu(x_0)y_0\right]$$

2.3.6

Theorem Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a,b) that contains the point x_0. Then for any choice of the initial value y_0, there exists a unique solution $y(x)$ on (a,b) to the initial value problem $\frac{dy}{dx} + P(x)y = Q(x)$, $y(x_0) = y_0$. In fact,

$$y(x) = \frac{1}{\mu(x)}\left[\int_{x_0}^{x} \mu(t)Q(t)\,dt + \mu(x_0)y_0\right]$$

where $\mu(x) = e^{\int P(x)\,dx}$.

Note: We will not memorize this solution. We will follow the following steps to solve 1st order linear ODE's.

To Solve a 1st Order Linear ODE

1. Write it in the form $\frac{dy}{dx} + P(x)\,y = Q(x)$.

2. Find the integrating factor $\mu(x) = e^{\int P(x)\,dx}$. Use $C = 0$.

3. Multiply both sides by $\mu(x)$ $\quad \mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$

4. The left side can be rewritten $\quad \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)Q(x)$

   Check that this is the case!

5. Integrate both sides $\quad \mu(x)y = \int \mu(x)Q(x)\,dx + C$

   and solve for y.

← 2.3.8

6. If given an initial condition, use it to find c.

7. If cannot integrate in part 5 and the initial condition is $y(x_0) = y_0$, then use a definite integral as follows

$$\mu(x) y = \int_{x_0}^{x} \mu(t) Q(t)\, dt + c$$

Solve for y and use $y(x_0) = y_0$ to find c.

<u>Ex</u> Solve $y' + \frac{2}{x} y = \frac{\sin x}{x^2}$

↓

2.3.9 $\rightarrow$

<u>Ex</u> Solve the IVP $y' - 2xy = x$, $y(0) = 1$

← 2.3.10

Ex Solve $\frac{dy}{dx} - 2xy = 1$, $y(0) = -\frac{1}{2}$.

Sec 2.4 Exact Equations

Consider the first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Suppose $F(x,y) = c$ it is general implicit solution.

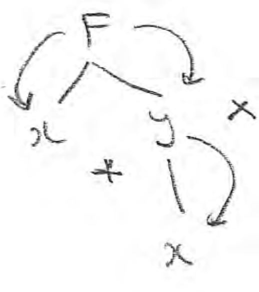
That is, $y = y(x)$ is the general solution where

$F(x, y(x)) = c.$

By Implicit Differentiation

$$\frac{d}{dx}F(x, y(x)) = \frac{d}{dx}c$$

By the general chain rule (Calc III),



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

← 2.4.2

Comparing the two equations

$$M + N \frac{dy}{dx} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0,$$

we must have $\frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$.

<u>Def</u> The differential form $M(x,y)\,dx + N(x,y)\,dy$, the ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ or the ODE in differential form $M(x,y)\,dx + N(x,y)\,dy = 0$ is called <u>exact</u> in a rectangle R in the plane if there is a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$ in R.

Note: Of course, the general implicit sol of an exact equation is $F(x,y) = C$.

Notice that for $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$, since the mixed 2nd order partial derivatives are equal to one another for "nice" functions, we have

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial M}{\partial y} = M_y$$

Equal Equal

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x} = N_x$$

<u>Test For Exactness</u> Consider ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \text{ or}$$

equivalently $M(x,y)dx + N(x,y)dy = 0$.

Suppose M, N, M_y, and N_x are continuous in

← 2.4.4

a rectangle R in the plane, then this ODE is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{or } M_y = N_x)$$

in R.

<u>Ex</u> Show that $\dfrac{dy}{dx} = -\dfrac{2y+3xy^2}{2x+3x^2y}$ is an exact equation.

<u>Theorem</u> The implicit general solution of the exact ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$(\text{or } M(x,y)\,dx + N(x,y)\,dy = 0)$$

is $F(x,y) = C$, where $\frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$.

<u>Ex</u> Solve the exact ODE

$$(2y + 3xy^2)\,dx + (1 + 2x + 3x^2y)\,dy = 0.$$

2.4.6

2.4.7 →

To Solve the Exact Eq $M + N\frac{dy}{dx} = 0$

1. Start with $\frac{\partial F}{\partial x} = M$ or $\frac{\partial F}{\partial y} = N$

Integrate both sides with respect to x or y, respectively. The constant of integration is a function of y or x, respectively:

$$F = \int M\,dx + g(y), \text{ or } F = \int N\,dy + h(x)$$

2. Take the partial derivative with respect to y in the 1st case or wrt x in the 2nd case, and set equal to N or M, recpectively.

$$\frac{\partial}{\partial y}\left(\int M\,dx + g(y)\right) = N \text{ or } \frac{\partial}{\partial x}\left(\int N\,dy + h(x)\right) = M$$

← 2.4.8

3. Solve for $g'(y)$ or $h'(x)$, respectively. Integrate wrt y or x to find g or h, respectively. Use $c=0$ for constant of integration.

4. The solution is $F(x,y)=C$ where

$F=\int M\,dx+g(y)$ or $F=\int N\,dy+h(x)$.

2.4.9 →

Ex Solve the IVP

$$(1+x+e^{x}\sin y+3y)+(3x+e^{x}\cos y)\frac{dy}{dx}=0,$$

$$y(0)=\frac{\pi}{2}.$$

2.4.10

2.4.10 →

Ex Solve $\dfrac{dy}{dx} = -\dfrac{y^3 + xy^2}{3xy^2 + x^2 y}$.

← 2.4.11

Note: What if the last problem was stated as

$$(y^2+xy)+(3xy^2+x^2)\frac{dy}{dx}=0\ ?$$

Sec 2.5 Special Integrating Factors

From the last section, we can solve the ODE

$$(y^2+xy)+(3xy^2+x^2)\frac{dy}{dx}=0$$

by 1st multiplying both sides by y.

So, given the ODE $M(x,y)+N(x,y)\frac{dy}{dx}=0$,

can we find an integration factor μ such that the eq

$$\mu M(x,y)+\mu N(x,y)\frac{dy}{dx}=0$$

is exact?

If the eq $\mu M+\mu N\frac{dy}{dx}=0$ is exact, then we must have

← 2.5.2

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

This equation may have no solution or many solutions! To solve it we limit ourselves to just two special cases.

<u>1st Case</u> Assume $\mu = \mu(x)$. That is, $\frac{\partial \mu}{\partial y} = \mu_y = 0$.

Then
$$\cancelto{0}{\frac{\partial \mu}{\partial y}} M + \mu \frac{\partial M}{\partial y} = \cancelto{\frac{d\mu}{dx}}{\frac{\partial \mu}{\partial x}} N + \mu \frac{\partial N}{\partial x}$$

$$N \frac{d\mu}{dx} = \frac{\partial M}{\partial y}\mu - \frac{\partial N}{\partial x}\mu$$

$$\frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\mu$$

Since $\frac{d\mu}{dx}$ and μ are functions of x only, this can only hold if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is also a function of x.

<u>Ex</u> Solve $(2y + 3xy^2)\,dx + (x + 2x^2y)\,dy = 0$

2.5.4

Consider the nonexact eq $M(x,y)+N(x,y)\frac{dy}{dx}=0$.

1. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is a function of x, then

$$\mu(x)M(x,y)+\mu(x)N(x,y)\frac{dy}{dx}=0$$

is an exact eq where $\mu=\mu(x)$ is a sol

of $\dfrac{d\mu}{dx}=\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}\mu$.

2. Similarly, if $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is a func of y, then

$$\mu(y)M(x,y)+\mu(y)N(x,y)\frac{dy}{dx}=0$$

is an exact eq where $\mu=\mu(y)$ is a sol

of $\dfrac{d\mu}{dy}=\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}\mu$.

← 2.5.6

Your book solves the equations of μ and state μ like we did for the linear equations, but it is a bit harder to write/remember.

In (1) $$\mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$$, while

in (2) $$\mu(y) = e^{\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy}$$!

<u>Ex</u> Solve $(y^2 + xy)\,dx + (3xy + x^2)\,\frac{dy}{dx} = 0$

2.5.7 →

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{(3y+2x)-(2y+x)}{y^2+xy}$$

←2.5.8

2.5.9 →

Ex Solve $(x+2)\sin y + x\cos y\,\frac{dy}{dx} = 0$.

$M = (x+2)\sin y \qquad N = x\cos y$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(x+2)\cos y - \cos y}{x\cos y} = $$

$$= \frac{x+1}{x}$$

$$\frac{d\mu}{dx} = \frac{x+1}{x}\mu \implies \int \frac{d\mu}{\mu} = \int \frac{x+1}{x}\,dx = \int \left(1+\frac{1}{x}\right)dx$$

← 2.5.10

2.5.11 →

There are many other types of μ can be considered.

Also, there can be more than one integrating factor.

Ex Show that $\mu(x) = x$ and $\mu(x,y) = \dfrac{1}{xy(2x+y)}$

are both integrating factors for the eq.

$$(3xy+y^2)+(x^2+xy)\frac{dy}{dx}=0.$$

2.5.12

Also, it is worthwhile to notice that the 1st order linear eq $\frac{dy}{dx} + P(x)y = Q(x)$ can be made exact using its integration factor $\mu = e^{\int P(x)\,dx}$. That is

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

is an exact equation.

$$\underbrace{\left[\mu(x)P(x)y - \mu(x)Q(x)\right]}_{M} + \underbrace{\mu(x)}_{N}\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = \mu(x)\,P(x)$$

← 2.5.14

$$\frac{\partial N}{\partial x} = \frac{d\mu}{dx} = \frac{d}{dx}\left[e^{\int P(x)\,dx}\right] = e^{\int P(x)\,dx} \frac{d}{dx}\left(\int P(x)\,dx\right)$$

$$= \mu(x)\, P(x)$$

so, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Finally, given a 1st order ODE we must recognize its solution technique and it is possible that it can be solved more than one way.

In Sec 2.6, your book covers other types of equations, in particular, homogeneous & Bernoulli equations, but we don't have time to cover them!

ERSE1 →

Equations Reducible to Separable Equations

This is a topic that is covered briefly in section 4.7 of your textbook. However, it is easier if we have a discussion of it sooner.

The 2nd order ODE's of the form

$$P(x)y'' + Q(x)y' = 0$$

can be reduced to 1st order separable ODE's, as follows. Let $u = y'$. Then $u' = y''$ and the equation reduces to $P(x)u' + Q(x)u = 0$, which is a separable ODE. After solving for

ERS E2

u, we can find y by solving $y' = u$, which is again a separable ODE.

Ex Solve $y'' - y' = 0$

Let $u = y'$. Then $u' = y''$.

$$y'' - y' = 0 \Rightarrow u' - u = 0 \Rightarrow \frac{du}{dx} = u$$

$$\int \frac{du}{u} = \int dx \Rightarrow \ln|u| = x + C \Rightarrow |u| = e^{x + C_1}$$

$$|u| = e^{C_1} e^{x} = c_1 e^{x} \Rightarrow u = \pm c_1 e^{x} \Rightarrow u = c_1 e^{x}$$

$$y' = u = c_1 e^{x} \Rightarrow y = \int c_1 e^{x} dx$$

$$y = c_1 e^{x} + c_2$$

Ex Solve $xy'' + y' = 0$.

Let $u = y'$. Then $u' = y''$.

$$xy'' + y' = 0 \Rightarrow xu' + u = 0 \Rightarrow x\frac{du}{dx} + u = 0$$

$$x\frac{du}{dx} = -u \Rightarrow \frac{du}{u} = -\frac{1}{x}\,dx$$

$$\int \frac{du}{u} = \int -\frac{1}{x}\,dx \Rightarrow \ln|u| = -\ln|x| + C_1 \Rightarrow$$

$$|u| = e^{-\ln|x| + C_1} = e^{C_1} e^{-\ln|x|} = C_1 e^{\ln|x|^{-1}} = C_1 |x|^{-1}$$

$$u = \pm C_1 \frac{1}{|x|} = C_1 \frac{1}{\pm x} = \pm C_1 \frac{1}{x} = C_1 \frac{1}{x}$$

$$y' = u = C_1 \frac{1}{x} \Rightarrow y = \int C_1 \frac{1}{x}\,dx$$

$$y = C_1 \ln|x| + C_2$$

ERSE 4

Ex Solve $xy'' + (x-1)y' = 0$. Assume $x > 0$.

Let $u = y'$. Then $u' = y''$.

$$xy'' + (x-1)y' = 0 \Rightarrow xu' + (x-1)u = 0 \Rightarrow$$

$$x\frac{du}{dx} = -(x-1)u \Rightarrow \frac{du}{u} = -\frac{(x-1)}{x}dx$$

$$\int \frac{du}{u} = \int -\frac{(x-1)}{x}dx = -\int\left(1-\frac{1}{x}\right)dx$$

$$\ln|u| = -(x - \ln|x|) + C_1 = -x + \ln x + C_1$$

$$|u| = e^{-x+\ln x + C_1} = e^{-x}e^{\ln x}e^{C_1} = C_1 x e^{-x}$$

$$u = \pm C_1 x e^{-x} = C_1 x e^{-x}$$

$$y' = u = C_1 x e^{-x} \Rightarrow y = \int C_1 x e^{-x} dx$$

$$y = C_1(-xe^{-x} - e^{-x}) + C_2$$ using integration by parts

$$y = -C_1(1+x)e^{-x} + C_2$$

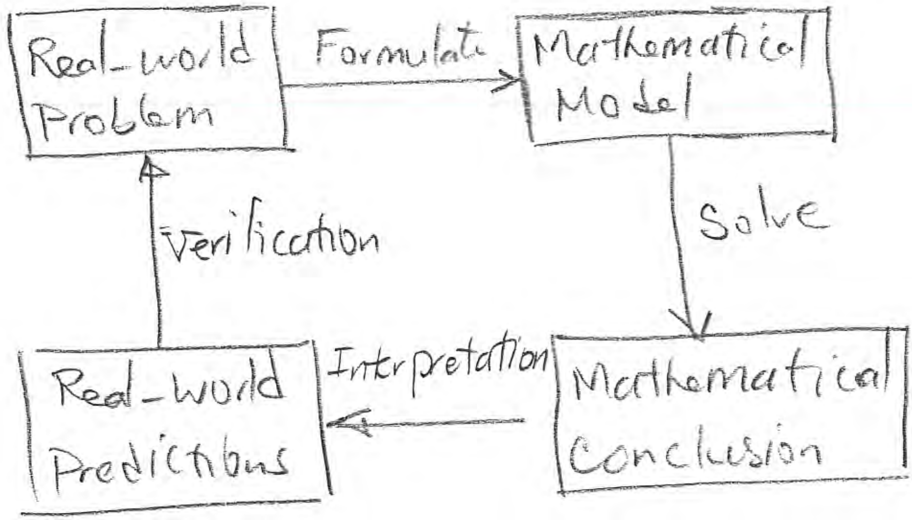
Chapter 3 Mathematical Models

Sec 3.1 Mathematical Modeling

Please read section 3.1. Here is a very brief description.

A mathematical model is never a completely accurate representation of a physical situation-it is an idealization. A good model simplifies reality enough to permit mathematical calculations but is accurate enough.

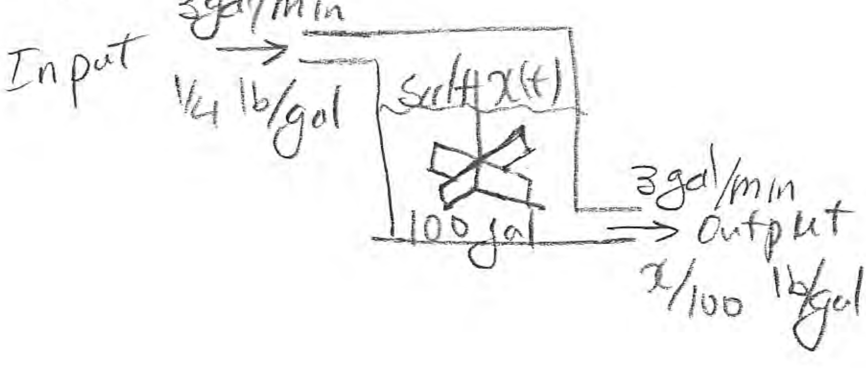
← 3.2.2



Sec 3.2 Compartmental Analysis

We will discuss two types of problems:
Mixing Problems & Population Models.

Ex At the time $t=0$, a tank contains 50 lb of salt dissolved in 100 gal of water. Assume water containing $\frac{1}{4}$ lb of salt per gallon is entering the tank at a rate of 3 gal/min, and that the well-stirred mixture is draining from the tank at the same rate. Find the amount of salt $x(t)$ in the tank at any time t, and also find the limiting amount x_L that is present after a very long time. Find the time after which the salt level is within 2% of x_L.



3.2.4

3.2.5 →

3.2.6

<u>Ex</u> Reconsider the last problem. Suppose the tank can hold 400 gallons and assume water containing $\frac{1}{4}$ lb of salt per gallon is entering at the rate of 6 gal/min, and the mixture still leaves at the rate of 3 gal/min. Find the amount of salt in the tank at the moment of overflow.

3.2.7 →

3.2.8

Population Models

Let $p(t)$ be the size of the population of a give species at time t.

Exponential Growth (Multusian) Model

It is assumed that population grows at a rate proportional to the population size present

$$\frac{dp}{dt} = rp \qquad r \text{ is called the (relative) growth rate}$$

$$p(0) = p_0$$

The solution is $p(t) = p_0 e^{rt}$.

This model is accurate for many populations, at least for a limited period of time. Obviously, the limitations of space, food supply, and others do not allow the population continue to grow at this rate.

Ex Assume Earth's population grows at a rate proportional to the current population. Using the data

Year	Pop size
1900	1 billion
1930	2 billion

develop a model of Earth's population. How long will it take for the Earth's population to double in size.

← 3.2.10

How good is this model? Earth's pop size in the year 2011 was 7 billion.

Doubling Time

A feature of the exponential growth (or decay) is a fixed doubling (halving) time T.

$$\begin{cases} \dfrac{dP}{dt} = rP \\ P(0) = P_0 \end{cases} \implies P(t) = P_0 e^{rt}$$

Find t for $P(t) = 2P_0$

← 3.2.12

Logistic Growth

It is assumed that the population growth rate itself depends on the current population size.

$$\frac{dp}{dt} = f(p)\,p$$

A simple logistic model is

$$\frac{dp}{dt} = r\left(1-\frac{p}{k}\right)p$$
$$p(0) = p_0$$

r is the intrinsic growth rate and k is the carrying capacity.

Notice that for $p<k$, $\frac{dp}{dt}>0$, while for $p>k$, $\frac{dp}{dt}<0$.

Your book's notation is $\frac{dp}{dt} = -A(p-p_1)\,p$.

In comparison, $A=\frac{r}{k}$ and $p_1 = k$.

3.2.13 →

<u>Ex</u> Solve the logistic model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{k}\right)P, \quad P(0) = P_0.$$

Show $\lim_{t \to \infty} P(t) = k$.

3.2.14

3.2.15 →

← 3.2.16

Ex Develop a logistic model for the Earth's population using the following data.

Year	Popsize
1900	1 billion
1930	2 billion
1960	3 billion

3.2.17 →

3.2.18

How good is this model? Earth's Pop size in the year 2011 was 7 billion.

Sec 3.4 Newtonian Mechanics

Newton's Law of Motion

Force = Mass x Acceleration

$$F = ma$$

Units of Force

Metric: Newton, N

Metric: Dyne, dyn

Brititsh Imperial: Pound, lb

$$1\,N = 1\,\frac{kg\,m}{sec^2}, \quad 1\,dyn = 1\,\frac{g\,cm}{sec^2}$$

$$1\,lb = 1\,\frac{slug\,ft}{sec^2}$$

weight = Mass x Acceleration due to gravity

$$w = mg$$

← 3.4.2

Units of weight: N (or dyn) and lb

Units of mass: kg and slug

$g = 9.81 \text{ (or } 9.8) \frac{m}{sec^2} = 32 \text{ (or } 32.17) \frac{ft}{sec^2}$

Acceleration is the rate of change of velocity:

$$a = \frac{dv}{dt}$$

$$F = ma = m\frac{dv}{dt}$$

Note: Colloquially, people say "I bought 2 kilograms of oranges" or "I weight 80 kilograms", when the correct unit of weight is Newton, not kilogram. So pay attention; if a problem state an object <u>weighs</u> 5 kilograms, it is actually 5 Newtons.

Motion of a (Falling) Body

Derivation of the Model

Consider an object of mass m which is moving up or down. It's weight always points down. The air resistance force is always in the opposite direction of movement and depends on the velocity of the object: f(v).

$$F = ma = m\frac{dv}{dt}$$

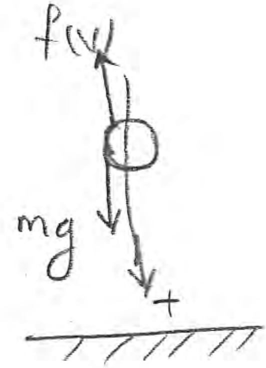
F = Sum of forces acting on the object (weight & air resistant)

← 3.4.4

Down direction positive

Object moving down

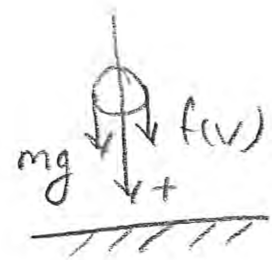
$$m\frac{dv}{dt} = F =$$



Down direction positive

Object moving up

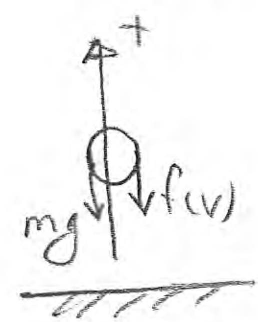
$$m\frac{dv}{dt} = F =$$



Up direction positive

Object moving up

$$m\frac{dv}{dt} =$$

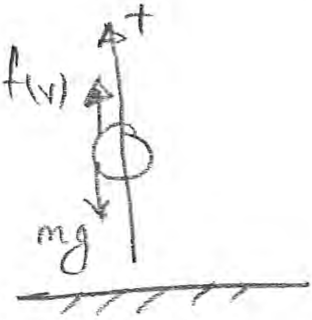


3.4.5 →

Up direction positive

Object moving down

$$m\frac{dv}{dt} =$$



Typical forms of $f(v)$ are $f(v) = b|v|$ & $f(v) = bv^2$.

$$|v| = \begin{cases} v, & \text{if } v \geq 0 \Leftrightarrow \text{Moving in the positive direction} \\ -v, & \text{if } v < 0 \Leftrightarrow \text{Moving in the opposite direction of positive direction} \end{cases}$$

Apply $f(v) = b|v|$ in the last four cases to

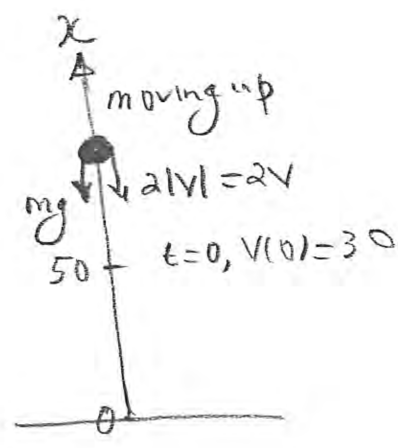
See that

Down direction positive: $m\frac{dv}{dt} = mg - bv$

Up direction positive: $m\frac{dv}{dt} = -mg - bv$

3.4.6

Ex A body of mass 10 kg is thrown upward with the initial velocity of 30 m/sec from a height of 50 meters. The air resistance for this body is twice its speed (magnitude of velocity). Find its maximum distance from the ground. Find the time it hits the ground.



$$m\frac{dv}{dt} = F =$$

$$\begin{cases} \dfrac{dv}{dt} + 0.2V = -9.81 \\ V(0) = 30 \end{cases}$$

3,4,7 →

$$V(t) = -49.05 + 79.05\, e^{-0.2t}$$

At the Max distance, $V(t) = 0$

$$-49.05 + 79.05\, e^{-0.2t} = 0$$

3.4.8

The object reaches its maximum distance from the ground after $t \simeq 2.39$ seconds.

$$\begin{cases} \frac{dx}{dt} = v = -49.05 + 79.05\, e^{-0.2t} \\ x(0) = 50 \end{cases}$$

$$x(t) = 445.25 - 49.05t - 395.25\, e^{-0.2t}$$

Max Distance $= x(2.39) =$

$= 82.9567\ldots \simeq 83$

The maximum distance from the ground is about 83 meters.

3.4.9 →

When the object hits the ground, $x(t)=0$

$$445.25 - 49.05t - 395.25e^{-0.2t} = 0$$

Can solve this using the Newton's method or Mathematica. Since we know this time is more that twice the time to reach the highest point, start with, say, $t_1 = 5$, in the Newton's method

Mathematica Notebook

The object hits the ground after $t \approx 7.15$ seconds

3.4.10

Note: If we knew the value of $395e^{-0.2t}$ is close to zero, for example, if we knew t value would be large, say $t=50$, $395e^{-0.2(50)} = -0.0179$, we could ignore this term and just solve the remaining linear equation.

It is also possible to find the maximum distance without finding x as a function of t (solving the second IVP). By the Chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$

$$-0.2v - 9.81 = \frac{dv}{dx}v$$

3.4.11 $\rightarrow$

$$\begin{cases} v \dfrac{dv}{dx} = -0.2v - 9.81 \\ \text{At } v = 30,\ x = 50 \end{cases}$$

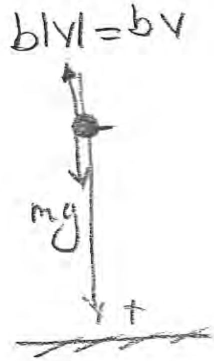
← 3.4.12

$$25(0.2v+9.81)-245.25\ln|0.2v+9.81|=-x-231.798$$

At the maximum distance, $v=0$!

3.4.13 →

A General Case



Down direction positive

Moving down

$f(|v|) = b|v|$

Initial velocity $v(0) = v_0$

$$\begin{cases} m\dfrac{dv}{dt} = mg - bv \\ v(0) = v_0 \end{cases}$$

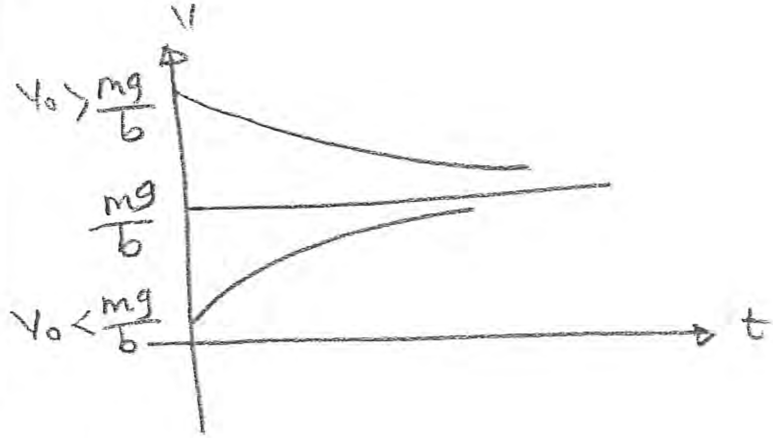
$$v(t) = \frac{mg}{b} + \left(v_0 - \frac{mg}{b}\right)e^{-\frac{b}{m}t}$$

Limiting Velocity

$$v_L = \lim_{t\to\infty} v(t) = \frac{mg}{b}$$

This implies that heavier objects fall faster!

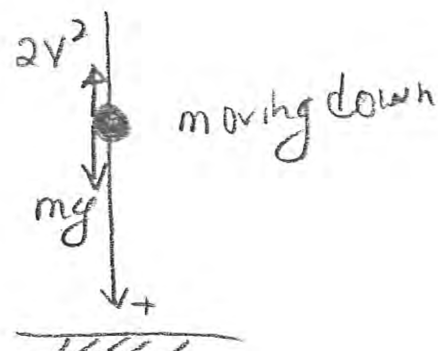
← 3.4.14



Ex A body of mass 1 slug is thrown downward with the initial velocity of 44 ft/sec in a medium offering resistance twice the square of the velocity. Find its limiting velocity.

$m \frac{dv}{dt} = F =$

$v(0) =$



3.4.15 $\rightarrow$

$$\begin{cases} \dfrac{dv}{dt} = 32 - 2v^2 \\ v(0) = 44 \end{cases}$$

3.4.16

$$v(t) = \frac{4ce^{16t} - 4}{1 + ce^{16t}}, \quad v(0) = 44$$

$$v(t) = \frac{-4.8e^{16t} - 4}{1 - 1.2e^{16t}}$$

Limiting Velocity $= V_L = \lim\limits_{t \to \infty} v(t) = \lim\limits_{t \to \infty} \frac{-4.8e^{16t} - 4}{1 - 1.2e^{16t}}$

$$= \lim_{t \to \infty} \underline{\qquad\qquad}$$

Chapter 4 Linear Second-Order Equations

Sec 4.1 Introduction: The Mass-Spring Oscillations

Please read section 4.1. We will derive the spring-mass-dashpot system later. You can easily do the problems in this section.

Section 4.2 Homogeneous Linear Equations: The General Solution

Your textbook covers a mixture of method of solutions (without proof) and theory in several sections of this chapter. I would like to cover a significant part of the theory in this section.

4.2.2

Theory of 2nd Order Linear ODE's

Def A 2nd order Lin. ODE is an eq. of the form

$$P(t)\, y'' + Q(t)\, y' + R(t)\, y = G(t)$$

A 2nd order Lin. homogeneous ODE is an eq. of the form

$$P(t)\, y'' + Q(t)\, y' + R(t)\, y = 0$$

A 2nd order Lin Homog. constant coefficient ODE is an eq. of the form

$$a y'' + b y' + c y = 0.$$

Notice that we are using t as the independent variable and $y = y(t)$.

We can rewrite the general 2nd order Lin. ODE as

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p(t) = \frac{Q(t)}{P(t)}$, $q(t) = \frac{R(t)}{P(t)}$, and $g(t) = \frac{G(t)}{P(t)}$.

Existence Theorem If functions $p(t)$, $q(t)$ and $g(t)$ are continuous on the open interval (a,b) which contains t_0. Then there exists a unique function $y = \phi(t)$ satisfying the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

on the interval (a,b).

← 4.2.4

Ex $y'' - 4t\,y' + (4t^2 - 2)y = e^{t^2}$

$y(0) = 1, \quad y'(0) = 1$

has a unique solution on the entire real number line $(-\infty, \infty)$ since $p(t) = -4t$, $q(t) = 4t^2 - 2$, and $g(t) = e^{t^2}$ are continuous on $(-\infty, \infty)$.

Def Let L be the differential operator

$$L(\phi) = \phi'' + p(t)\phi' + q(t)\phi.$$

Note: Operator is used when a functions acts on other function; a func whose domain is a set of function. Also, sometimes an operator is written using brackets $L[\phi]$.

L is just a func!

Ex $L(t^2+1) = (t^2+1)'' + p(t)\,(t^2+1)' + q(t)\,(t^2+1)$

$$= 2 + p(t)(2t) + q(t)\,(t^2+1)$$

Note: We can also write L as

$$L = D^2 + pD + q$$

In this version the argument of L is not written, that is, $L(\phi) = D^2\phi + pD\phi + q\phi$, where $D\phi = \phi'$ and $D^2\phi = \phi''$.

Notice that the 2nd order Lin. ODE's can be written as

$$y'' + p(t)y' + q(t)y = g(t) \Rightarrow L(y) = g(t)$$

$$y'' + p(t)y' + q(t)y = 0 \Rightarrow L(y) = 0$$

← 4.2.6

<u>Superposition Principle</u> If y_1 and y_2 are two solutions of $L(y)=0$, then their linear combination $y = c_1 y_1 + c_2 y_2$, with c_1 & c_2 constants, is also a solution.

<u>Proof</u>

$$LHS = L(y) = L(c_1 y_1 + c_2 y_2) =$$

$$= (c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + p(t)(c_1 y_1' + c_2 y_2') + c_1 q(t) y_1 + c_2 q(t) y_2$$

$$= c_1 y_1'' + c_1 p(t) y_1' + c_1 q(t) y_1 + c_2 y_2'' + c_2 p(t) y_2' + c_2 q(t) y_2$$

$$= c_1 (y_1'' + p(t) y_1' + q(t) y_1) + c_2 (y_2'' + p(t) y_2' + q(t) y_2)$$

$$= c_1 L(y_1) + c_2 L(y_2) = c_1 0 + c_2 0 = 0 = RHS$$

4.27 $\longrightarrow$

Note: In above we have shown L is a linear operator!

Def A function (or operator) $f(x)$ is called linear if $f(c_1x_1+c_2x_2)=c_1f(x_1)+c_2f(x_2)$ for c_1, c_2 constants and x_1 & x_2 in the domain of f.

Ex $f(x)=3x$ is a linear func since

$$f(c_1x_1+c_2x_2)=3(c_1x_1+c_2x_2)$$
$$=c_1(3x_1)+c_2(3x_2)$$
$$=c_1f(x_1)+c_2f(x_2)$$

Ex Show $f(x)=1-x^2$ is not a linear function

We just need to show

4.2.8

$f(c_1x_1+c_2x_2) \neq c_1f(x_1)+c_2f(x_2)$ for one

choice of c_1, c_2, x_1 and x.

Here is one such choice. Let $c_1=c_2=1$, $x_1=1$ &

$x_2=5$.

$$f(c_1x_1+c_2x_2) = f(x_1+x_2) = f(1+5) = f(6)$$
$$= 1-6^2 = 1-36 = -35$$

$$c_1f(x_1)+c_2f(x_2) = f(x_1)+f(x_2) = f(1)+f(5)$$
$$= (1-1^2)+(1-5^2)$$
$$= 0-24 = -24$$

So, $f(c_1x_1+c_2x_2) \neq c_1f(x_1)+c_2f(x_2)$

Hence, $f(x)=1-x^2$ is not a linear function.

Note: The only real-valued linear func is $f(x) = mx$

for any constant m.

Ex Show that $L(\phi) = \phi'' + p(t)\phi' + q(t)\phi$ is a

linear operator.

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)'$$
$$+ q(t)(c_1 y_1 + c_2 y_2)$$

$$= (c_1 y_1'' + c_2 y_2'') + p(t)(c_1 y_1' + c_2 y_2')$$
$$+ c_1 q(t) y_1 + c_2 q(t) y_2$$

$$= c_1 (y_1'' + p(t) y_1' + q(t) y_1)$$
$$+ c_2 (y_2'' + p(t) y_2' + q(t) y_2)$$

$$= c_1 L(y_1) + c_2 L(y_2)$$

Thus, L is a linear operator.

4.2.10

Def Two functions f and g are said to be linearly dependent on the interval I if for two constants k_1 and k_2, not both zero,

$k_1 f(x) + k_2 g(x) = 0$ for all x in I.

Two func that are not linearly dependent are called linearly independent.

Two functions f and g are said to be linearly independent on the interval I if for some constants k_1 and k_2

$k_1 f(x) + k_2 g(x) = 0$ for all x in I, then

$k_1 = k_2 = 0$.

Ex Show that functions $f(x) = 3-9x$ and $g(x) = 6x-2$ are linearly dependent.

It is easy to see that

$$2f(x) + 3g(x) = 2(3-9x) + 3(6x-2)$$
$$= 0 \quad \text{for all } x$$

Note: An easy way to check if two functions are lin. dependent is to check if one is a constant multiple of the other. In the above example, $g(x) = -\frac{2}{3} f(x)$.

Ex Show that functions $f(x) = \sin x$ and $g(x) = \cos x$ are linearly independent on the interval $I = (-\infty, \infty)$.

← 4.2.12

Suppose $k_1 \sin x + k_2 \cos x = 0$ for all x in $(-\infty, \infty)$. We need to show $k_1 = k_2 = 0$. Since $k_1 \sin x + k_2 \cos x = 0$ for all x, I will choose convenient x values.

For $x = 0$, $k_1 \sin 0 + k_2 \cos 0 = 0 \implies k_2 = 0$

For $x = \frac{\pi}{2}$, $k_1 \sin \frac{\pi}{2} + k_2 \cos \frac{\pi}{2} = 0 \implies k_1 = 0$

Note: An easy way to check if <u>two</u> functions are linearly independent is to check that one of them is not a constant multiple of the other.

Ex Show the given pair of functions are linearly independent on the interval $(-\infty, \infty)$.

1. $f(x) = e^{-x}$, $g(x) = e^{2x}$
2. $f(x) = 1$, $g(x) = x$

1. Since $e^{-x} \neq c\, e^{2x}$ for any constant c, f & g are Lin. independent.
2. Since $1 \neq cx$ for any constant c, f & g are Lin. independent.

Def The Wronskian of two differentiable functions $f(x)$ and $g(x)$ is defined as

$$W(f, g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x).$$

4.2.14

EX Find the Wronskian of $f(x) = \sin x$ and $g(x) = \cos x$.

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x$$
$$= -1$$

EX Find the Wronskinan of $f(x) = e^{-x}$ and $g(x) = e^{2x}$.

$$W(e^{-x}, e^{2x}) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = e^{-x}(2e^{2x}) - e^{2x}(-e^{-x})$$
$$= 2e^{x} + e^{x} = 3e^{x}$$

Note: We may write $W(f,g)$ as just W to shorten it or as $W(f,g)(x)$ to emphasis the independent variable.

Also, notice that W is an operator (acts on functions), so we could write it as $W[f,g](x)$.

Theorem Let $p(t)$ and $q(t)$ be continuous on the open interval (a,b). Suppose $y_1(t)$ and $y_2(t)$ are two solutions of the linear homogeneous equation $L(y) = y'' + p(t)y' + q(t)y = 0$, on (a,b). Then

$$W(y_1, y_2) = C\, e^{\int_a^t -p(s)\,ds}$$ for some constant C.

Proof $L(y_1) = 0$ & $L(y_2) = 0$. So

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

$$(y_1 y_2'' - y_2 y_1'') + p(t)(y_1 y_2' - y_2 y_1') = 0$$

4.2.16

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$. Notice that

$$W' = y_1' y_2' + y_1 y_2'' - (y_2' y_1' + y_2 y_1'')$$

$$= y_1 y_2'' - y_2 y''$$

So, our last equation is $W' + p(t) W = 0$

$$\frac{dW}{dt} = -p(t) W$$

$$\int \frac{dW}{W} = \int -p(t)\,dt$$

$$\ln|W| = e^{\int_a^t -p(z)\,dz} + C$$

$$|W| = e^{c}\, e^{\int_0^t -p(z)\,dz} = C e^{\int_a^t -p(z)\,dz}$$

$$W = \pm C e^{\int_a^t -p(z)\,dz} = C e^{\int_a^t -p(z)\,dz}$$

Note: This theorem implies that either W is never zero or is always zero since exponential functions are never zero.

4.2.17 →

<u>Theorem</u> let $p(t)$ and $q(t)$ be continous on the open interval (a,b). Suppose $y_1(t)$ and $y_2(t)$ are two solutions of the linear homogeneous equation $L(y) = y'' + p(t)y' + q(t)y = 0$, on (a,b). Then $y_1(t)$ and $y_2(t)$ are linearly independent on (a,b) if and only if $W(y_1, y_2)(t) \neq 0$ for at least one $t = \tau$ value in (a,b).

<u>Proof</u> (i) Assume $W(y_1, y_2)(\tau) \neq 0$. Show y_1 & y_2 are linearly independent. We will prove this by contradiction. If y_1 & y_2 are linearly dependent, then $y_1(t) = k y_2(t)$ for some constant k. Then $y_1'(t) = k y_2'(t)$ and so

← 4.2.18

$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = k y_2 y_2' - y_2 k y_2' = 0$ for all t.

So, $W(y_1, y_2)(\tau) = 0$. A contradiction. Hence, y_1 & y_2 are linearly independent.

(ii) Assume y_1 & y_2 are linearly independent. Show $W(y_1, y_2)(t) \neq 0$ for some $t = \tau$ value. We will also prove this by contradiction. Suppose $W(y_1, y_2)(t) = 0$ for all t values. Then $y_1(t) y_2'(t) - y_2(t) y_1'(t) = 0$ for all t values. Now,

$$\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \frac{y_2' y_1 - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2} = 0 \Rightarrow$$

$$\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = 0 \Rightarrow \frac{y_2}{y_1} = c \Rightarrow y_2 = c\, y_1$$

That is, y_1 & y_2 are linearly dependent. A contradiction. So, $W(y_1, y_2)(t) \neq 0$ for at least one $t = \tau$ value.

4.2.19 →

<u>Theorem</u> Let $p(t)$ and $q(t)$ be continuous on the open interval (a,b). Suppose $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the linear homogeneous equation $L(y) = y'' + p(t)y' + q(t)y = 0$. Then, every solution of $L(y)=0$ is of the form $c_1 y_1(t) + c_2 y_2(t)$ for some constants c_1 and c_2.

<u>Proof</u> Since y_1 & y_2 are Lin. Indep Sol of $L(y)=0$, there is a number τ such that $W(y_1, y_2)(\tau) \neq 0$. Let $\phi(t)$ be a solution of $L(y)=0$. Let $y_0 = \phi(\tau)$ and $y_1 = \phi'(\tau)$. We know the solution of

$$L(y)=0, \quad y(\tau) = y_0 \cdot \quad y'(\tau) = y_1$$

4.2.20

is unique. If we show $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a sol of this IVP, then we must have $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$.

We know that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a sol of $L(y) = 0$ for all constants c_1 and c_2. Now, we find c_1 and c_2 so that $y(t_0) = y_0$ and $y'(t_0) = y_1$.

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \Rightarrow c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$y'(t) = c_1 y_1'(t) + c_2 y_2'(t) \Rightarrow c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_1$$

4.2.21 $\rightarrow$

$$\begin{cases} c_1 y_1(\tau) + c_2 y_2(\tau) = y_0 \\ c_1 y_1'(\tau) + c_2 y_2'(\tau) = y_1 \end{cases}$$

$$c_1 = \frac{y_0 y_2'(\tau) - y_1 y_2(\tau)}{y_1(\tau) y_2'(\tau) - y_2(\tau) y_1'(\tau)} = \frac{y_0 y_2'(\tau) - y_1 y_2(\tau)}{W(y_1, y_2)(\tau)}$$

$$c_2 = \frac{y_1 y_1(\tau) - y_0 y_1'(\tau)}{y_1(\tau) y_2'(\tau) - y_2(\tau) y_1'(\tau)} = \frac{y_1 y_1(\tau) - y_0 y_1'(\tau)}{W(y_1, y_2)(\tau)}$$

Both c_1 & c_2 are well-defined since $W(y_1, y_2)(\tau) \neq 0$.

So $y(t) = c_1 y_1(t) + c_2 y_2(t)$ with c_1 & c_2 as above is the sol of the IVP. By the uniqueness of the sol $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$

← 4.2.22

We have shown that the general sol of

$$y'' + p(t)y' + q(t)y = 0$$

is $y(t) = C_1 y_1(t) + C_2 y_2(t)$ where $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of this ODE.

The lin. indep. sol y_1 and y_2 are called fundamental solutions. The set $\{y_1(t), y_2(t)\}$ is called a fundamental set of solutions.

4.2.23 →

<u>Theorem</u> $y=e^{rt}$ is a sol of $ay''+by'+cy=0$

where $ar^2+br+c=0$.

<u>Proof</u> $y=e^{rt}$, $y'=re^{rt}$, $y''=r^2e^{rt}$

$$LHS = ay''+by'+cy = ar^2e^{rt}+bre^{rt}+ce^{rt}$$
$$= (ar^2+br+c)e^{rt} = (0)e^{rt} = 0 = RHS$$

<u>Def</u> The characteristic equation of the 2nd order linear homogeneous ODE $ay''+by'+cy=0$ is $ar^2+br+c=0$.

Other names for the characteris equation are auxiliary (and indicial) equations

4.2.24

<u>Theorem</u> If $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ with $r_1 \neq r_2$ are two solutions of $ay'' + by' + cy = 0$, then y_1 & y_2 are linearly independent.

<u>Proof</u> Since y_1 & y_2 are solutions of this ODE, to show they are Lin. Indep. we just need to show $W(y_1, y_2)(T) \neq 0$ for some T value.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} r_2 e^{r_2 t}$$

$$- e^{r_2 t} r_1 e^{r_1 t} = (r_2 - r_1) e^{r_1 t} e^{r_2 t}$$

$\neq 0$ for any t value since $r_1 \neq r_2$ and exponential functions are never zero.

Theorem The general solution of $ay''+by'+cy=0$ is $y(t)=c_1 y_1(t)+c_2 y_2(t)$ where $y_1(t)=e^{r_1 t}$, $y_2(t)=e^{r_2 t}$ and r_1 and r_2 are two (real-valued) unequal solutions of the characteristic equation $ar^2+br+c=0$.

Proof Obvious!

General Solution of $ay''+by'+cy=0$

Characteristic (auxiliary) Equation

$$ar^2+br+c=0$$

Sol $$r_1, r_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

4.2.26

(a) The characteris eq has two real and unequal solutions r_1 & r_2 or $b^2 - 4ac > 0$.

Gen sol $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

(b) The characteristic eq has repeated real roots; $b^2 - 4ac = 0$ and $r_1 = r_2 = -\frac{b}{2a}$.

Gen sol $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

(c) The characteristic eq has two complex conjugate solutions $r_1, r_2 = \alpha \pm i\beta$ with $\beta \neq 0$. This occurs when $b^2 - 4ac < 0$.

Gen sol $y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$

4.2.27 →

We have proven part a of the above. Parts b and c are covered in later sections and will be proven later.

<u>Ex</u> Find the general solution of $y'' - 2y' - 5y = 0$

<u>Auxiliary Eq:</u>

4.2.28

Ex Find the solution of $2y'' + 3y' = 0$, $y(0) = 0$

and $y'(0) = 1$.

Characteristic Eq:

4.2.29 $\rightarrow$

<u>Ex</u> Find the solution of $y'' - 3y = 0$, $y(0) = 1$, and $y'(0) = 0$.

← 4.2, 30

<u>Ex</u> Find the solution of $2y'' + y' - y = 0$, $y(1) = 0$,

$y'(1) = 1$.

4.2.31 →

Ex Find the general solution of $9y'' - 12y' + 4y = 0$

<u>Aux. Eq.</u>

$\leftarrow$ 4.2.32

<u>Ex</u> Find the sol of $y'' + 4y' + 4y = 0$, $y(0) = 1$, and $y'(0) = 0$.

Sec 4.3 Auxiliary Equations with Complex Roots

Consider the 2nd order linear homogeneous ODE with constant coefficients

$$a y'' + b y' + c y = 0.$$

Suppose characteristic roots are complex valued.

$$a r^2 + b r + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{with } b^2 - 4ac < 0$$

Then $r = \alpha \pm i\beta$ where $\alpha = -\frac{b}{a}$ & $\beta = \frac{\sqrt{4ac - b^2}}{2a}$.

We know $y = e^{(\alpha \pm i\beta)t}$ are solutions of this ODE. But, do you recall what is $y = e^{(\alpha \pm i\beta)t}$?

4.3.2

Euler's Formula $\quad e^{i\theta} = \cos\theta + i\sin\theta$

Note: The derivative of the complex-valued function is

$$y = f(x) + i\,g(x) \text{ is } y' = f'(x) + i\,g'(x).$$

So, $$e^{(\alpha - i\beta)t} = e^{\alpha t - i\beta t} = e^{\alpha t} e^{i(-\beta t)}$$

$$= e^{\alpha t}\left(\cos(-\beta t) + i\sin(-\beta t)\right)$$

$$= e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$$

$$= e^{\alpha t}\cos(\beta t) - i\,e^{\alpha t}\sin(\beta t)$$

$$e^{(\alpha + i\beta)t} = e^{\alpha t + i\beta t} = e^{\alpha t} e^{i(\beta t)}$$

$$= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$$

$$= e^{\alpha t}\cos(\beta t) + i\,e^{\alpha t}\sin(\beta t)$$

Since $y_1 = e^{(\alpha - i\beta)t} = e^{\alpha t}\cos(\beta t) - i e^{\alpha t}\sin(\beta t)$ and

$y_2 = e^{(\alpha + i\beta)t} = e^{\alpha t}\cos(\beta t) + i e^{\alpha t}\sin(\beta t)$ are

solutions of the ODE, any linear combination of

them is also a solution.

So, $z_1 = \frac{1}{2} y_1 + \frac{1}{2} y_2 =$

and $z_2 = \frac{i}{2} y_1 - \frac{i}{2} y_2 =$

4.3.4

are also solutions. Now, we show $z_1 = e^{\alpha t}\cos\beta t$ and $z_2 = e^{\alpha t}\sin\beta t$ are linearly independent.

$$W(z_1, z_2)(t) = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix}$$

$=$

$= e^{\alpha t}\cos\beta t\,($

$\quad - e^{\alpha t}\sin\beta t\,($

$= \beta e^{2\alpha t}\cos^2\beta t + \beta e^{2\alpha t}\sin^2\beta t$

$=$ $\qquad\qquad = \beta e^{2\alpha t} \neq 0$

Since $\beta \neq 0$ & $e^{2\alpha t} \neq 0$

Thus, $z_1 = e^{\alpha t}\cos\beta t$ and $z_2 = e^{\alpha t}\sin\beta t$ are two linearly independent solutions of $ay'' + by' + cy = 0$.

Hence, the general sol of $ay'' + by' + cy = 0$ with complex conjugate characteristic roots

$$ar^2 + br + c = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \alpha \pm i\beta$$

where $b^2 - 4ac < 0$ $\left(\alpha = -\frac{b}{2a}, \ \beta = \frac{\sqrt{4ac - b^2}}{2a} \right)$ is

$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t.$$

This proves part c of Gen. Sol of $ay'' + by' + cy = 0$ which was stated in the last section.

← 4.3.6

Ex Find the general solution of $y'' - 2y' - 3y = 0$.

4.3.7 →

Ex Find the solution of $y''+9y=0$, $y(0)=1$, $y'(0)=1$.

4.3.8

Sections 4.4 & 4.5 Nonhomogeneous Equations –

The Method of Undetermined Coefficients

We want to solve 2nd order linear nonhomogeneous equations

$$y'' + p(t)\,y' + q(t)\,y = g(t).$$

Recall that $L(y) = y'' + p(t)y' + q(t)y$.

Theorem If y_1 and y_2 are two solutions of the nonhomogeneous equation

$$L(y) = y'' + p(t)y' + q(t)y = g(y).$$

Then $y_1 - y_2$ is a solution of the homogeneous equation $L(y) = 0$.

4.4/5.2

Proof Recall that L is a linear operator. Then

$$L(y_1 - y_2) = L(y_1) - L(y_2) = g(y_1) - g(y_2) = 0$$

Thus, $y_1 - y_2$ is a solution of $L(y) = g(y)$.

This means that the general solution of $L(y) = g(t)$ is of the form $y = y_h + y_p$ where y_h is a solution of the homogeneous equation $L(y) = 0$ and y_p is one solution of the nonhomogeneous equation $L(y) = g(t)$. y_p is called a particular solution.

Theorem If y_f is a solution of $L(y) = f(t)$ and y_g is a solution of $L(y) = g(t)$, then $y = y_f + y_g$ is a solution of $L(y) = f(t) + g(t)$, where $L(y) = y'' + p(t)y' + q(t)y$.

Proof Recall that L is a linear operator. Then $L(y_f + y_g) = L(y_f) + L(y_g) = f(t) + g(t)$. Hence $y = y_f + y_g$ is a sol of $L(y) = f(t) + g(t)$.

This implies that in order to find a particular sol y_p of $L(y) = f(t) + g(t)$, find particular solutions y_f of $L(y) = f(t)$ and y_g of

4.4/5.4

$L(y) = g(t)$, and add them, $y_p = y_f + y_g$.

How to find a particular solution

Method of Undetermined Coefficients

This method works for ODE's with constant coefficients $ay'' + by' + cy = g(t)$ with $g(t)$ of certain types!

4.4/5.5 →

Ex Find the general solution of $y''+3y'+2y=t^2-2$.

Gen Sol $y=y_h+y_p$ where y_h is the general solution of the Homog. Eq. $y''+3y'+2y=0$ and y_p is a solution of $y''+3y'+2y=t^2-2$

Homog. Sol $y''+3y'+2y=0$

Charac. Eq.

$$y_h=C_1e^{-2t}+C_2e^{-t}$$

Now, let's look for a particular solution by making an educated guess at it!

4.4/5.6

$$y'' + 3y' + 2y = t^2 - 2$$

Let's look for a particular solution of the form

$$y_p = At^2 + Bt + C$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' + 3y_p' + 2y_p = \overset{\text{want}}{=} t^2 - 2$$

↓

$$y_p = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{3}{4}$$

Gen Sol $\quad y = y_h + y_p$

$$y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{3}{4}$$

Ex Find the general solution of $y'' + 3y' = t^2 - 2$.

Homog Eq. $\quad y'' + 3y' = 0$

$$y_h = c_1 + c_2 e^{-3t}$$

Look for a particular solution of the form

$$y_p = At^2 + Bt + C$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

4.4/5.8

$$y_p'' + 3y_p' = \quad \overset{\text{want}}{=} t^2 - 2$$

No such particular solution! The problem is that a constant c is a solution of the homogeneous eq.
We need a higher degree polynomial. Let try

$$y_p = t(At^2 + Bt + C) = At^3 + Bt^2 + Ct$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' + 3y_p' = \quad \overset{\text{want}}{=} t^2 - 2$$

↓

So, $y_p = \frac{1}{9}t^3 - \frac{1}{9}t^2 - \frac{16}{27}t$

<u>Gen. Sol</u>

$$y = y_h + y_p$$

$$y = C_1 + C_2 e^{-3t} + \frac{1}{9}t^3 - \frac{1}{9}t^2 - \frac{16}{27}t$$

4.4/5.10

The Particular Solution of $ay'' + by' + cy = g(t)$

$g(t)$	$y_p(t)$
$P_n(t) = a_n t^n + \dots + a_1 t + a_0$	$t^s(A_n t^n + \dots + A_1 t + A_0)$
$P_n(t)\, e^{rt}$	$t^s(A_n t^n + \dots + A_0)\, e^{rt}$
$P_n(t)\, e^{\alpha t} \begin{cases} \sin\beta t \\ \text{or} \\ \cos\beta t \end{cases}$	$t^s\left[(A_n t^n + \dots + A_0)\, e^{\alpha t}\cos\beta t + (B_n t^n + \dots + B_0)\, e^{\alpha t}\sin\beta t\right]$

where s is the smallest nonnegative integer ($s = 0, 1$ or 2) which will ensure that no term in y_p is a solution to the corresponding homogeneous equation. Equivalently, for the above three cases, respectively, s in the number of times 0 is a root of the auxiliary eq, r is a root of the auxiliary eq, and $\alpha + i\beta$ is a root of the auxiliary equation.

4.4/5.11 →

Ex Find a particular solution of $y'' + 3y' + 2y = 5 + xe^{2x}$.

<u>Homog. Sol</u>

$$y'' + 3y' + 2y = 0$$
$$r^2 + 3r + 2 = 0$$
$$(r+2)(r+1) = 0$$
$$r+2 = 0, \quad r+1 = 0$$
$$r = -2, \quad r = -1$$
$$y_1 = e^{-2t}, \quad y_2 = e^{-t}$$
$$y_h = c_1 e^{-2t} + c_2 e^{-t}$$

<u>Particular Sol</u> Consider

$y'' + 3y' + 2y = 5$ and $y'' + 3y' + 2y = xe^{2x}$

$y_{p_1} =$

$y_{p_2} =$

So, $y_p = y_{p_1} + y_{p_2} =$

← 4.4/5.12

4.4/5.13 →

<u>Ex</u> Find the solution of $y'' + 2y' + y = e^{x}\cos x$,

$y(0) = y'(0) = 0$

4.4/5.14

4.4/5.15 $\rightarrow$

Ex Determine a suitable form of y_p for the following equations.

1. $y'' - 5y' + 6y = e^{x}\cos 2x + (x^2+1)e^{2x}$

1.4/5.16

2. $y'' - 5y' + 6y = e^{x} \cos 2x + (x^2+1) e^{2x} \sin x$

4.4/5. 17 →

3. $y'' + 2y' + 2y = 3e^{-x} + 2e^{-x}\cos x + 4x^2 e^{-x}\sin x$

4.45.18

Sec 4.6 Variation of Parameters

In this lecture we cover the method of variation of parameters which is covered in section 4.6 and part of section 4.7. We used the method of Undetermined Coefficients to find a particular solution of ODE's of the form

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is of a certain type. What if $g(t)$ is not of that type and/or coefficients of the ODE are not constants.

Consider $\underbrace{y'' + p(t)\,y' + q(t)\,y}_{L(y)} = g(t)$

4.6.2

Its general solution is $y = y_h + y_p$ where y_h is the general solution of the homogeneous equation $L(y)=0$ and y_p is a solution of $L(y)=g(t)$. The general sol of $L(y)=0$ is

$$y_h = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are two linearly independent solutions of $L(y)=0$. We look for a particular solution of the form $y_p = v_1 y_1 + v_2 y_2$ where v_1 & v_2 are functions to be determined. Let plug in y_p in the eq. $L(y)=g$. The problem we will run into is having two unknowns v_1 & v_2 but just one equation. In order to find a unique sol, we introduce a convenient equation: $v_1' y_1 + v_2' y_2 = 0$.

Now, we want to find v_1 and v_2 such that $y_p = v_1 y_1 + v_2 y_2$ is a solution of $L(y) = g(t)$ and $v_1' y_1 + v_2' y_2 = 0$. Let's plug in y_p in the eq.

$L(y) = g(t)$.

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' = v_1 y_1' + v_2 y_2'$$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

$$y_p'' + p(t) y_p' + q(t) y_p = g(t)$$

4.6.4

$$\left(y_1'' + p(t)\,y_1' + q(t)\,y_1\right) v_1 + \left(y_2'' + p(t)\,y_2' + q(t)\,y_2\right) v_2$$
$$+ v_1' y_1' + v_2' y_2' = g(t)$$

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = g(t) \end{cases}$$

$$v_1' = \frac{-y_2 g}{y_1 y_2' - y_2 y_1'} = \frac{-y_2 g}{W(y_1, y_2)}$$

$$v_2' = \frac{y_1 g}{W(y_1, y_2)}$$

Integrate each to find v_1 & v_2. Then $y_p = v_1 y_1 + v_2 y_2$.

<u>Ex</u> Find a particular solution and the general solution of $y'' + 4y' + 4y = t^2 e^{-t}$, $t > 0$.

<u>Homog. Eq.</u> $y'' + 4y' + 4y = 0$

← 4.6.6

$W(y_1, y_2) =$

$$v_1' = \frac{-y_2 g}{W(y_1, y_2)} =$$

$v_1 =$

$$v_2' = \frac{y_1 g}{W(y_1, y_2)} =$$

$v_2 =$

4.6.7 $\longrightarrow$

$$y_p = v_1 y_1 + v_2 y_2$$

We can pick $y_p = -e^{-2t}\ln t$.

<u>Gen. Sol</u> $\quad y = y_h + y_p$

$y =$

4.6.8

The Method of Variation of Parameters

A particular solution of $y''+p(t)y'+q(t)y=g(t)$ is

$$y_p = v_1 y_1 + v_2 y_2$$

where $v_1' = \dfrac{-y_2 g}{W(y_1,y_2)}$, $v_2' = \dfrac{y_1 g}{W(y_1,y_2)}$, and y_1 & y_2

are two linearly independent solutions of the homogeneous equation $y''+p(t)y'+q(t)y=0$.

Ex $y_1 = e^{x^2}$ and $y_2 = x e^{x^2}$ are two linearly independent solutions of the homogeneous equation $y''-4xy'+(4x^2-2)y=0$. Find the solution of

$y''-4xy'+(4x^2-2)y = e^{x^2}$, $y(0)=1$, $y'(0)=1$.

4.6.9 →

$y_p = v_1 y_1 + v_2 y_2$, $v_1' = \dfrac{-y_2 g}{W(y_1, y_2)}$, $v_2' = \dfrac{y_1 g}{W(y_1, y_2)}$

$g = e^{x^2}$, $y_1 = e^{x^2}$, $y_2 = x e^{x^2}$

← 4.6.10

<u>Ex</u> Find the general solution of $y''+y=\tan t$, 4.6.11 —

$0<t<\pi/2$.

← 4.6.12

4.6.13 →

The general solution of $y'' + p(t)y' + q(t)y = g(t)$ is

$$y = c_1 y_1 + c_2 y_2 + y_1 \int \frac{-y_2 g}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g}{W(y_1, y_2)}\, dt.$$

If the initial conditions are $y(t_0) = y_0$, $y'(t_0) = y_1$

and we cannot evaluate the above integrals, we can

use

$$y = c_1 y_1 + c_2 y_2 + y_1 \int_{t_0}^{t} \frac{-y_2(s) g(s)}{W(y_1, y_2)(s)}\, ds + y_2 \int_{t_0}^{t} \frac{y_1(s) g(s)}{W(y_1, y_2)(s)}\, ds$$

4.6.14

Sec 4.7 Variable-Coefficient Equations

We have already covered the method of Variation of Parameters. Your textbook also discusses the Cauchy-Euler equation which we will cover later. The method of variation of parameter is dependent on finding two linearly independent solutions of the corresponding equations. But, we can only solve such equations with constant coefficients. Here, we can improve this situtation. Given one solution y_1 of the eq $y'' + p(t)y' + q(t)y = 0$, we can find a 2nd Lin. Indep. y_2 of this equation.

4.7.2

Method of Reduction of Order

Given one solution y_1 of $y'' + p(t) y' + q(t) y = 0$.
Find a 2nd linearly independent solution of it.
Look for a 2nd sol of the form $y(t) = V(t)\, y_1(t)$.

Ex Find a second linearly independent solution of
$t^2 y'' - t(t+2) y' + (t+2) y = 0$ if $y_1(t) = t$ is one
solution of it.

Let $y = V(t)\, y_1 = tV$

$y' =$

$y'' =$

$$t^2y'' - t(t+2)y' + (t+2)y = 0$$

$$t^3v'' - t^3v' = 0 \Rightarrow v'' - v' = 0.$$

I covered solving these type of equation in the Module called Equations Reducible to Separable Equations. Although this equation can be solved using the characteristic equation, we will use the more general method discussed in the earlier module.

Let $u = v'$. Then $u' = v''$.

$$v'' - v' = 0 \Rightarrow u' - u = 0 \Rightarrow \frac{du}{dt} - u = 0$$

$\leftarrow$ 4.7.4

$$\frac{du}{dt} = u \Rightarrow \int \frac{du}{u} = \int dt$$

$$v' = u = c_1 e^t \Rightarrow v = c_1 e^t + c_2$$

$$y = v y_1 = (c_1 e^t + c_2) t$$

We can pick the 2nd linearly independent solution to be $y_2 = t e^t$ since we already have $y_1 = t$ as a solution.

<u>Ex</u> Find a second linearly independent solution of $y'' - 4xy' + (4x^2 - 2)y = 0$ if $y_1(x) = e^{x^2}$ is one solution of it.

4.7.5 →

Let $y = v y_1 = e^{x^2} v$

$y' =$

$y'' =$

$$y'' - 4xy' + (4x^2 - 2)y = 0$$

← 4.7.6

$$2e^{x^2}v + 4x^2e^{x^2}v + 4xe^{x^2}v' + e^{x^2}v''$$
$$-8x^2e^{x^2}v - 4xe^{x^2}v'$$
$$+4x^2e^{x^2}v - 2e^{x^2}v = 0$$

$$e^{x^2}v'' = 0 \Rightarrow v'' = 0$$
$$v' = c_1$$
$$v = c_1x + c_2$$

$$y = vy_1 = (c_1x + c_2)e^{x^2} = c_1xe^{x^2} + c_2e^{x^2}$$

We can pick $y_2 =$ since e^{x^2} is a solution of the equation.

4.7.7 →

Derivative of the Formula for the 2nd Linearly Independent Solution.

Consider $y'' + p(t)y' + q(t) = 0$. Suppose $y_1(t)$ is one solution of it, $(y_1(t) \neq 0)$

Let $y = vy_1$

$$y'' + p(t)y' + q(t)y = 0$$

$$(y_1'' + p(t)y_1 + q(t)y_1)v + (2y_1' + p(t)y_1)v' + y_1 v'' = 0$$

$$y_1 v'' + (2y_1' + p(t)y_1)v' = 0$$

4.7.8

Let $u = v'$. Then $u' = v''$.

$$y_1 u' + (2y_1' + p(t)y_1)u = 0$$

$$\frac{du}{dt} + \left(p(t) + 2\frac{y_1'}{y_1}\right)u = 0$$

$$\frac{du}{u} = -\left(p(t) + 2\frac{y_1'}{y_1}\right)dt$$

$$\int \frac{du}{u} = -\int p(t)\,dt - 2\int \frac{y_1'}{y_1}\,dt$$

Let $z = y_1$

Then $dz = y_1'\,dt$

$$\ln|u| = -\int p(t)\,dt - 2\int \frac{1}{z}\,dz$$

$$\ln|u| = -\int p(t)\,dt - 2\ln|z| + c_1$$

$$|u| = e^{-\int p(t)\,dt - 2\ln|y_1| + c_1}$$

$$= e^{c_1} e^{-2\ln|y_1|} e^{-\int p(t)\,dt}$$

$$= c_1 e^{\ln|y_1|^{-2}} e^{-\int p(t)\,dt}$$

$$= c_1 y_1^{-2} e^{-\int p(t)\,dt}$$

4.7.9—

$$u = \pm c_1 y_1^{-2} e^{-\int p(t)\,dt} = c_1 y_1^{-2} e^{-\int p(t)\,dt}$$

$$v' = u = c_1 \frac{e^{-\int p(t)\,dt}}{y_1^2}$$

$$v = c_1 \int \frac{e^{-\int p(t)\,dt}}{y_1^2}\,dt + c_2$$

$$y = v y_1 = y_1 \left(c_1 \int \frac{e^{-\int p(t)\,dt}}{y_1^2}\,dt + c_2 \right)$$

We can choose

$$y_2 = y_1 \int \frac{e^{-\int p(t)\,dt}}{y_1^2}\,dt.$$

4.7.10

General solution of $ay'' + by' + cy = 0$ when

$b^2 - 4ac = 0$.

Charac. Eq. $ar^2 + br + c = 0$

So, $y = e^{r_1 t} = e^{-\frac{b}{2a}t}$ is one solution of this equation.

A 2nd Lin. Indep. Sol is

$$y_2 = y_1 \int \frac{e^{-\int p(t)\,dt}}{y_1^2}\,dt = e^{-\frac{b}{2a}t} \int \frac{e^{-\int \frac{b}{a}\,dt}}{\left(e^{-\frac{b}{2a}t}\right)^2}\,dt$$

$$= \qquad = e^{-\frac{b}{2a}t}(t + c)$$

$$= t e^{-\frac{b}{2a}t} + c e^{-\frac{b}{2a}t}$$

We can choose $y_2(t) = t\, e^{-\frac{b}{2a}t}$.

Ex Show $y_1(t) = e^{-\frac{b}{2a}t}$ and $y_2(t) = t\, e^{-\frac{b}{2a}t}$ are two linearly independent solutions of $ay'' + by' + cy = 0$, where $b^2 - 4ac = 0$. Moreover, the general solution of this ODE is

$$y_1 = c_1 e^{-\frac{b}{2a}t} + c_2 t\, e^{-\frac{b}{2a}t}.$$

4.7.12

This finishes the proof of the general solution of

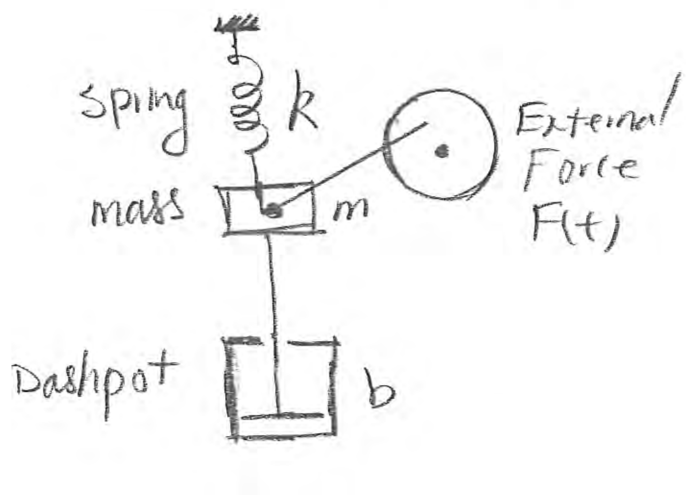
$ay'' + by' + cy = 0$

<u>Ex</u> Find a second linearly independent solution of $t^2 y'' + 6ty' + 6y = 0$ if $y_1(t) = t^{-2}$ is one solution of it. Assume $t > 0$.

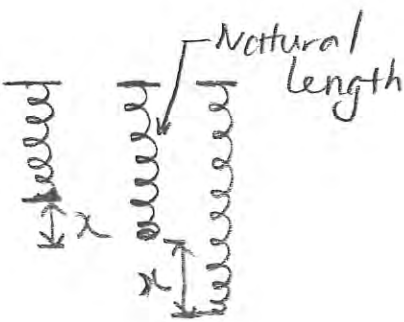
4.7.14

Sec 4.9 Mechanical Vibrations

Now, we discuss a spring-mass-dashpot (or damping) system. Your book covers a horizontal system and motions an equivalent vertical system. For the sake of completeness, I will discuss a vertical system in details and then mention the equivalent horizontal system. Here is a vertical system.



4.9.2

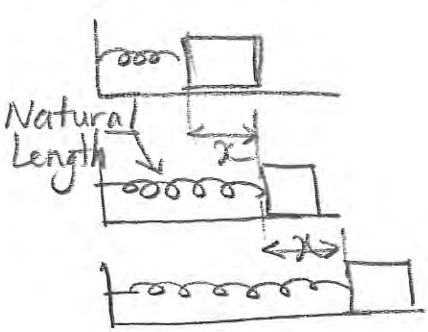


Hooke's Law The force required to stretch or compress a spring a distance of x from its natural length is $F_s = kx$.

	Units		
x	cm	m	ft
k	dyne/cm	N/m	lb/ft
F_s	dyne	N	lb

k is called the spring constant, $k>0$.

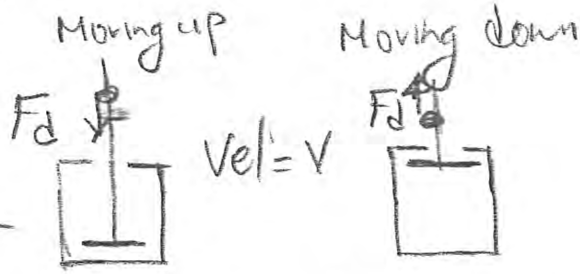
Of course, the same holds if stretching or compression is done horizontally



$F_s = kx$

Dashpot or Damping Force

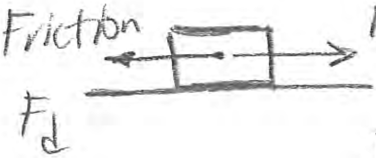
The damping force is proportional to the magnitude of the velocity (or speed) and acts in the opposite direction of motion.

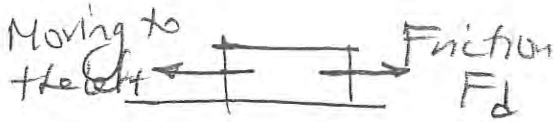


$F_d = b\,|V|$

b: Damping constant

The same is true if the damping force is the friction



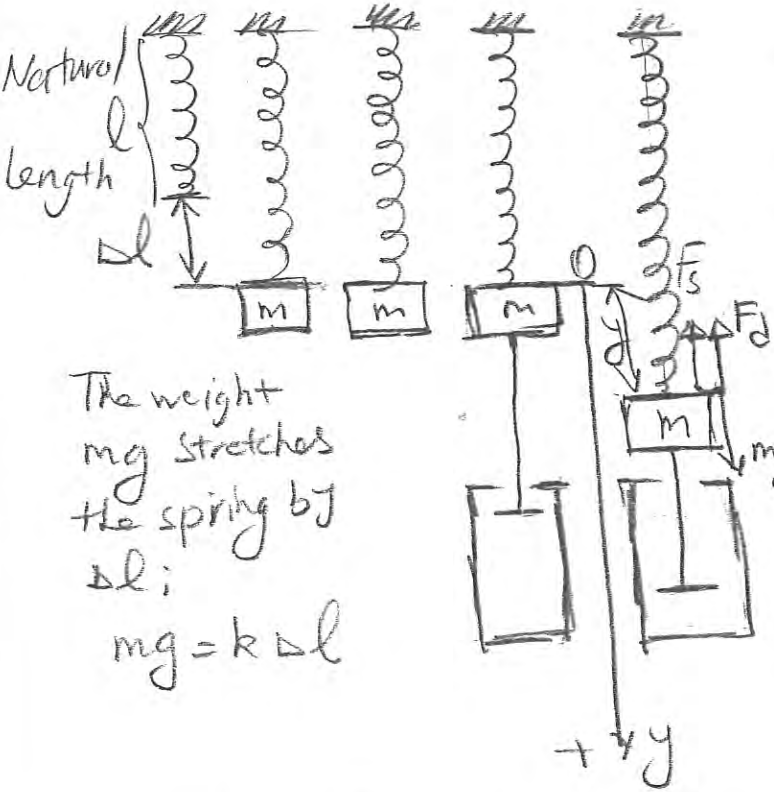


$F_d = b\,|V|$

b is the friction coefficient

← 4.94

	units		
$\lvert v \rvert$	cm/sec	m/sec	ft/sec
b	$\frac{\text{dyne-sec}}{\text{cm}}$	$\frac{\text{N-sec}}{\text{m}}$	$\frac{\text{lb-sec}}{\text{ft}}$
F_d	dyne	N	lb



The weight mg stretches the spring by $\Delta\ell$;

$$mg = k\Delta\ell$$

Suppose the spring-mass-dashpot is moving down.

$$\text{vel} = v = \frac{dy}{dt} > 0$$

$$\text{Total Force} = ma$$

$$F = m\frac{dv}{dt}$$

$$F = -F_s - F_d + mg$$

$$F_s = k(y + \Delta\ell), \quad F_d = b\frac{dy}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$$

4.95 →

$$F = m\frac{dv}{dt}$$

$$-F_s - F_d + mg = m\frac{d^2y}{dt^2}$$

$$-k(y+\Delta\ell) - b\frac{dy}{dt} + mg = m\frac{d^2y}{dt^2}$$

$$-ky - b\frac{dy}{dt} = m\frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

Initial Values

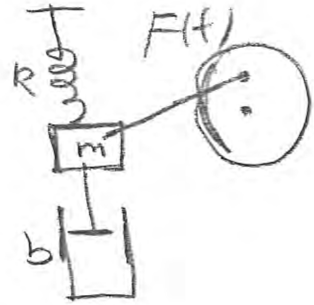
$y(0)$ initial position

$y'(0)$ initial velocity

This equation holds when down direction is the positive direction.

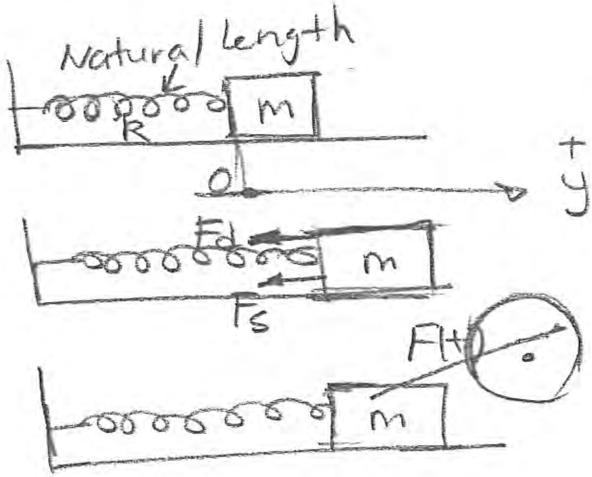
If the external force is $F(t)$,

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F(t)$$



4.9.6

In the horizontal situation



Right direction the positive direction

Moving to the right

Daming is the friction force and is to the left.

External force is F(t).

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F(t)$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F(t)$$

External Force
dyne, N
lb

mass
g
kg
slug

damping or friction constant
$\frac{\text{dyne-sec}}{\text{cm}}$, $\frac{\text{N-sec}}{\text{m}}$
$\frac{\text{lb-sec}}{\text{ft}}$

Spring Constant (stiffness)
dyne/cm, N/m
lb/ft

4.9.7 →

Ex State the IVP for the following spring-mass-dashpot system. It is known that a 10 lb weight stretches the spring 1 inch. The dashpot mechanism exerts a force of 1 pound for a velocity of 20 in/sec. A mass of 2 slugs is attached to the spring and simply released from a position $\frac{1}{6}$ ft above the equilibrium position. Note: Pay attention to units.

4.9.8

4.9.9 →

Ex State the IVP for the following system. A 2-kg mass is attatched to a spring with stiffness $k = 50$ N/m. The damping constant is 2 $\frac{\text{N-sec}}{\text{m}}$. The mass is pushed 50 cm to the left of equilibrium and given a leftward velocity of 2 m/sec.

← 4.9.10

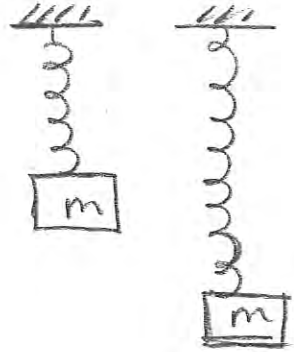
4.9.11 →

Free Vibration (External Force $F(t) = 0$)

$$my'' + by' + ky = 0$$

Undamped Free Vibration ($b = 0$ & $F(t) = 0$)

$$my' + ky = 0$$



$$y = c_1 \cos wt + c_2 \sin wt$$

where $w = \sqrt{\frac{k}{m}}$

w (omega) is the angular frequency with units $1/\text{time}$ (or rad/sec).

4.9.12

The period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ with units time.

The natural frequency is $\frac{1}{\text{period}} = \frac{\omega}{2\pi}$ with units cycles/time.

Recall that

$$y = A\cos\omega t + B\sin\omega t$$
$$= C\sin(\omega t + \phi)$$

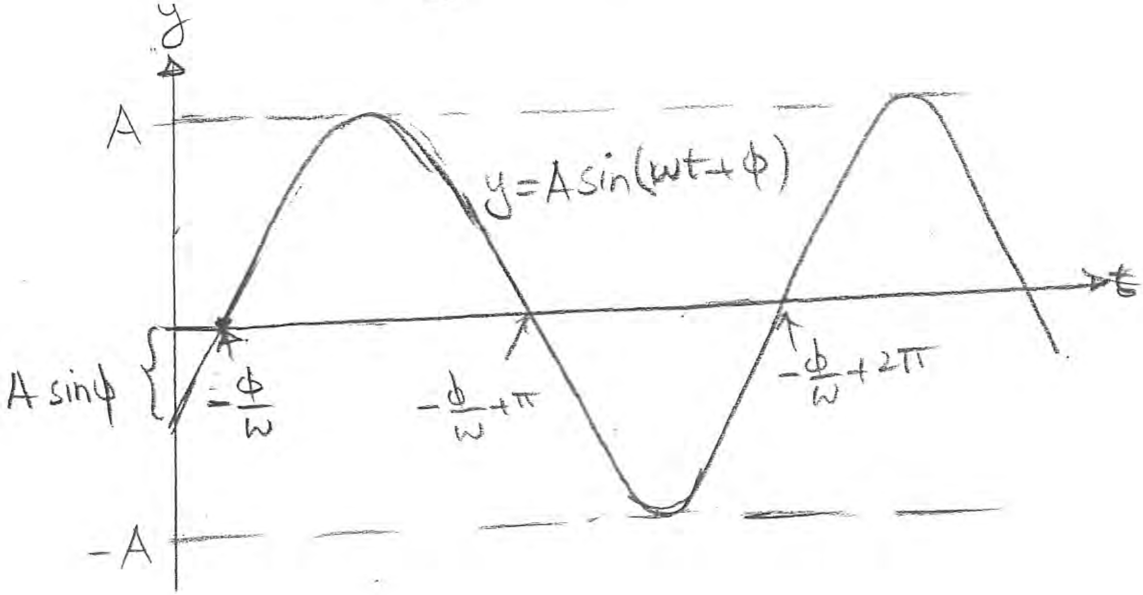
where $C = \sqrt{A^2+B^2}$, $\sin\phi = \frac{A}{C}$ and $\cos\phi = \frac{B}{C}$, or

$$\phi = \begin{cases} \tan^{-1}\left(\frac{A}{B}\right), & \text{if } B > 0 \\ \pi + \tan^{-1}\left(\frac{A}{B}\right), & \text{if } B < 0 \end{cases}$$

$$y = c_1 \cos wt + c_2 \sin wt = A \sin(wt + \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$, $\sin\phi = \frac{c_1}{A}$, $\cos\phi = \frac{c_2}{A}$ or

$$\phi = \begin{cases} \tan^{-1}\frac{c_1}{c_2}, & \text{if } c_2 > 0 \\ \pi + \tan^{-1}\frac{c_1}{c_2}, & \text{if } c_2 < 0 \end{cases}$$



Simple Harmonic Motion

← 4.9.14

Amplitude $= A$

Phase shift $= -\dfrac{\phi}{|\omega|} = -\dfrac{\phi}{\omega}$

If the phase shift is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ move the graph

of $y = A \sin \omega t$ to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$ by $|\text{phase shift}|$.

4.9.15 →

Ex A mass weighing 2 lb can stretch a 2-ft spring six inches. If the mass is pushed up by 3 in and put in motion with a downward velocity of 4 ft/sec, determine the subsequent motion. Find it natural frequency and the 1st time it passes the equilibrium position.

← 4.9.16

Damped, Free Vibration $(b \neq 0, F(t) = 0)$

$$my'' + by' + ky = 0, \quad m, b, k > 0$$

Charac. Eq. $mr^2 + br + k = 0$

Charac. Root $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

(i) $b^2 - 4mk > 0$.

$$0 \le \sqrt{b^2 - 4mk} < b \Rightarrow \quad r_1 = \frac{-b - \sqrt{b^2 - 4mk}}{2m} < 0$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} < 0$$

Gen Sol $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ with $r_1, r_2 < 0$

(ii) $b^2 - 4mk = 0$. $r = -\dfrac{b}{2m} < 0$

Gen Sol $y = c_1 e^{-\frac{b}{2m}t} + c_2 t e^{-\frac{b}{2m}t}$

← 4.9.18

(iii) $b^2 - 4mk < 0$. Let $\beta = \frac{\sqrt{4mk - b^2}}{2m}$

$$r = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm i\frac{\sqrt{4mk - b^2}}{2m}$$

$$= -\frac{b}{2m} \pm i\beta$$

$$y = c_1 e^{-\frac{b}{2m}t} \cos\beta t + c_2 e^{-\frac{b}{2m}t} \sin\beta t$$

$$= e^{-\frac{b}{2m}t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

$$= e^{-\frac{b}{2m}t}(A \sin(\beta t + \phi))) = A e^{-\frac{b}{2m}t} \sin(\beta t + \phi)$$

Ex Show that a spring-mass-dashpot system with no external force returns to the equilibrium position ($y \to 0$ as $t \to \infty$).

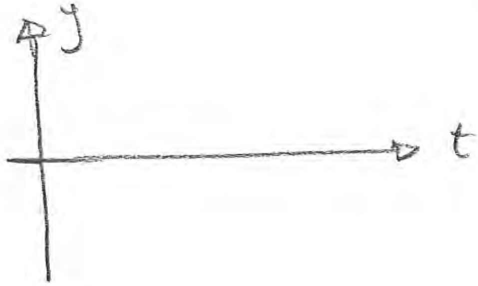
4.9.19 →

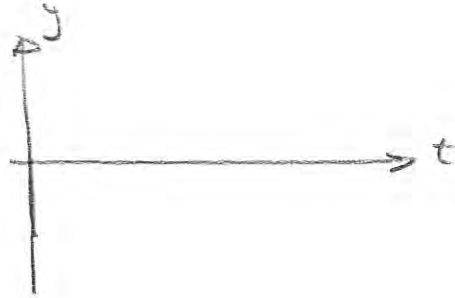
← 4.9.20

4.9.21 →

Case i ($b^2 - 4mk > 0$) is called overdamped.

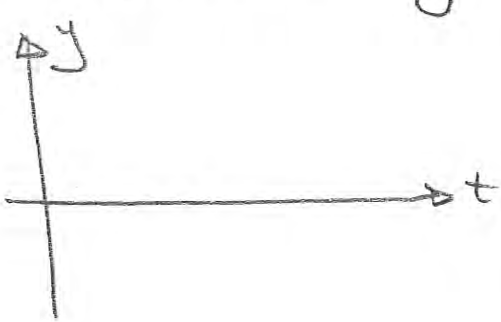
The system moves toward the equilibrium position without crossing it.

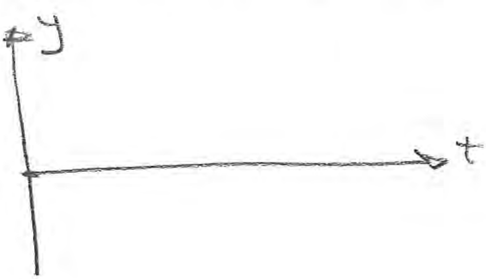




Case ii ($b^2 - 4mk = 0$) is called critically damped.

The system "creeps" back to its equilibrium position, but it may overshoot it once.

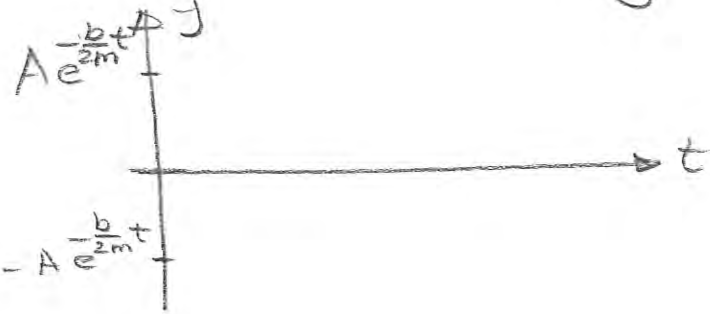




4.9.22

Case iii ($b^2-4mk<0$) is called underdamped.

The system moves toward eventual equilibrium position in an oscillatory fashion.



Ex Consider a spring-mass-dashpot with $m=1$ kg, $b=\frac{1}{8}\ \frac{N\text{-}sec}{m}$ & $k=1$ N/m. If the spring is pulled down 2 meters and let go, determine its subsequent motion. Determine the 1st time the mass passes the equilibrium position.

4.9.23 →

4.9.24

$$y = \frac{32}{\sqrt{255}} e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{255}}{16} t + \phi\right)$$

Quasi Period $T_d = \frac{2\pi}{\sqrt{255}/16} = \frac{32\pi}{\sqrt{255}} \approx 6.295$ sec

Quasi Freq $= \frac{1}{T_d} \approx 0.1588$ # of cycles per second

4.9.25 →

<u>Case (iii)</u> $b^2 - 4mk < 0$

$$y = c_1 e^{-\frac{b}{2m}t} \cos\beta t + c_2 e^{-\frac{b}{2m}t} \sin\beta t$$

$$= A\, e^{-\frac{b}{2m}t} \sin(\beta t + \phi)$$

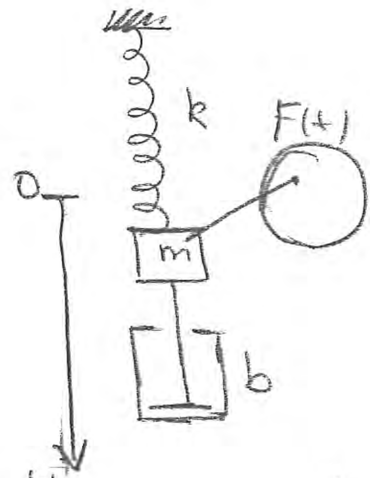
where $\beta = \dfrac{\sqrt{4mk - b^2}}{2m}$

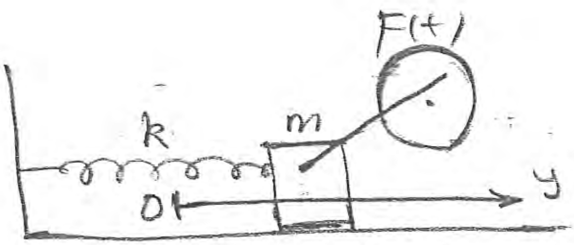
Quasi Period: $T_d = \dfrac{2\pi}{\beta}$ sec

Quasi Freq $= \dfrac{1}{T_d} = \dfrac{\beta}{2\pi}$ # of cycles per second

4.9.26

Sec 4.10 Forced Vibrations





Friction Coefficient = b

$$my'' + by' + ky = F(t), \quad F(t) \neq 0$$

Forced Vibration with no Damping ($b=0$, $F(t)\neq 0$)

$$my'' + ky = F(t)$$

We will consider the case $F(t) = F_0 \cos \gamma t$, with $\gamma > 0$

4.10. 2

Homog Eq. $my'' + ky = 0$

Homog Sol. $y_h(t) = c_1 \cos\omega t + c_2 \sin\omega t$ with

$$\omega = \sqrt{\frac{k}{m}}.$$

Particular Sol

① $\gamma \neq \omega$, $y_p(t) = A\cos\gamma t + B\sin\gamma t$

4.10.3 →

$$y_p(t) = \frac{F_0}{m(w^2-\gamma^2)} \cos\gamma t$$

<u>Gen sol</u> $y = y_h + y_p$

$$y(t) = c_1 \cos wt + c_2 \sin wt + \frac{F_0}{m(w^2-\gamma^2)} \cos\gamma t$$

<u>Ex</u> Consider a spring-mass system with external force $F = F_0 \cos\gamma t$, mass m and spring constant k and $\gamma \neq \sqrt{\frac{k}{m}}$. If the system is initially at rest, describe its subsequent motion

← 4.10.4

$$y = \frac{F_0}{m(\omega^2 - \gamma^2)} (\cos \gamma t - \cos \omega t)$$

4.10.5 →

Recall that

$$\cos x - \cos y = 2 \sin \frac{y-x}{2} \sin \frac{y+x}{2}$$

$$y = \underbrace{\left[\frac{2F_0}{m(\omega^2-\gamma^2)} \sin \frac{(\omega-\gamma)t}{2} \right]}_{\text{sinusoidal amplitude}} \sin \frac{(\omega+\gamma)t}{2}$$

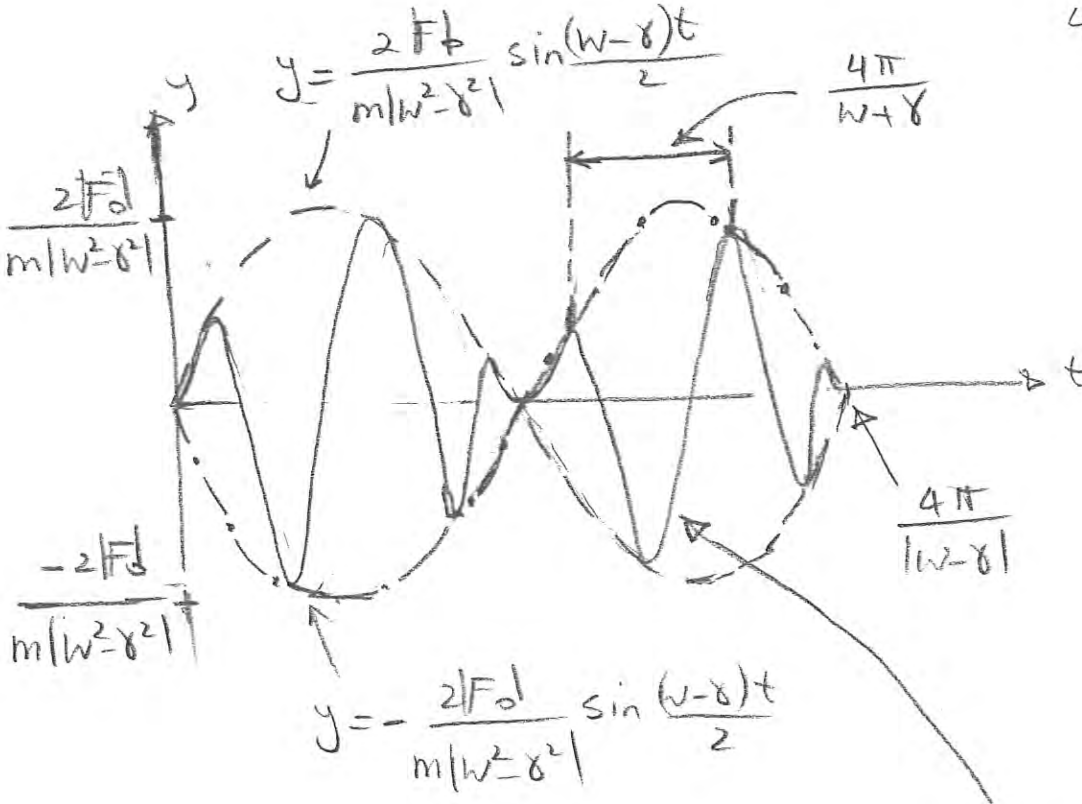
← 4.10.6

$\sin\frac{(\omega+\gamma)t}{2}$ completes one cycle every $\frac{2\pi}{\frac{\omega+\gamma}{2}} = \frac{4\pi}{\omega+\gamma}$ seconds.

$\sin\frac{(\omega-\gamma)t}{2}$ completes one cycle every $\frac{2\pi}{\left|\frac{\omega-\gamma}{2}\right|} = \frac{4\pi}{|\omega-\gamma|}$ seconds.

So, $\sin\frac{(\omega+\gamma)t}{2}$ oscillates much faster than $\sin\frac{(\omega-\gamma)t}{2}$.

4.10.7 →



Sol: $y = \dfrac{2F_0}{m(w^2-\gamma^2)} \sin\dfrac{(w-\gamma)t}{2} \sin\dfrac{(w+\gamma)t}{2}$

This type of motion is called a beat.

← 4.10.8

Ex A mass weighing 4 lb stretches a spring 1.5 in. The mass is displaced 2 inches in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted upon by an external force of $2\cos 3t$ lb, find the position of mass at any time t.

4.10.9 →

4.10.10

4.10.11 →

$$my'' + ky = F_0 \cos \gamma t$$

$$y_h = c_1 \cos wt + c_2 \sin wt \, , \quad w = \sqrt{\frac{k}{m}}$$

Particular Sol

(2) $\gamma = w = \sqrt{\frac{k}{m}}$, $y_p = t(A\cos\gamma t + B\sin\gamma t)$

$$= t(A\cos wt + B \sin wt)$$

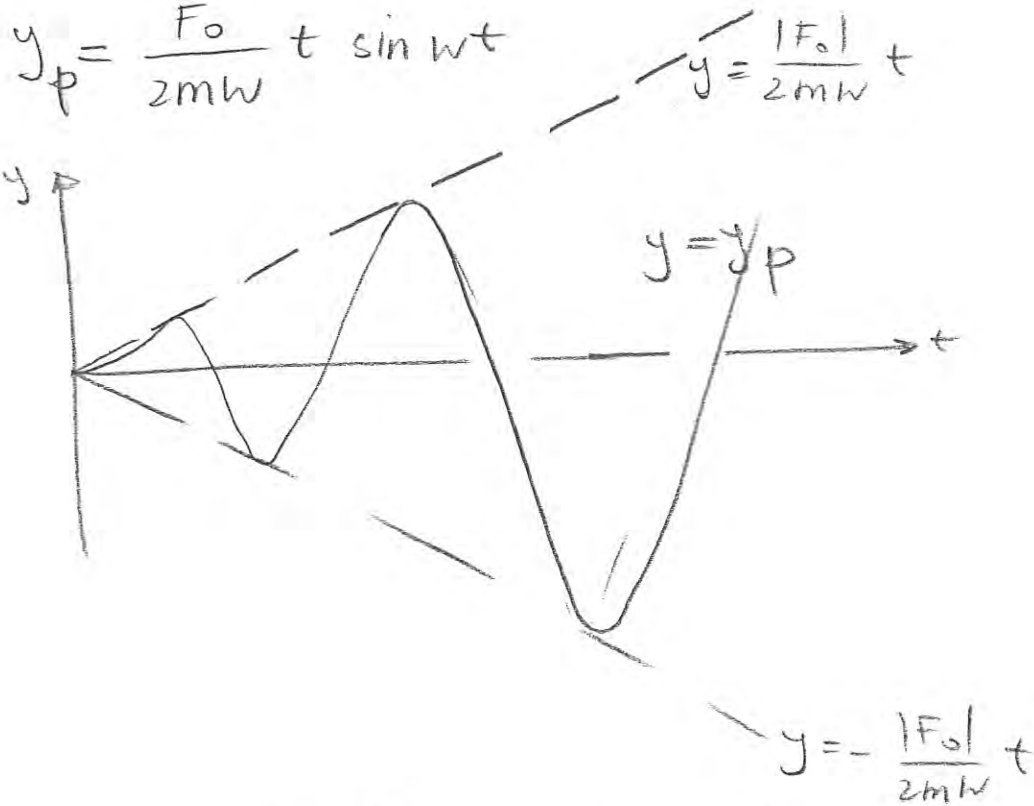
← 4.10.12

$$y_p(t) = \frac{F_0}{2mw} t \sin wt$$

$$y = C_1 \cos wt + C_2 \sin wt + \frac{F_0}{2mw} t \sin wt$$

4.10.13 →

$$y_p = \frac{F_0}{2mw} t \sin wt$$



The amplitude of $y = y_h + y_p$ will increase as t increases, and the system will tear itself apart as t increases. This is called <u>resonance</u>.

4.10.14

The resonance in $my'' + ky = F_0 \cos \gamma t$, $\gamma > 0$, occurs if $\gamma = \omega = \sqrt{\frac{k}{m}}$. That is, the period of the forcing function is the same as the natural period of the system.

<u>Ex</u> In the last example if $F(t) = 2\cos \gamma t$,

resonance occurs if $\gamma =$

4.10.15 →

Forced Vibration with Damping

$$my'' + by' + ky = \underbrace{F(t)}_{F_0 \cos \gamma t}, \quad b \neq 0, \ F(t) \neq 0$$

Homog Eq. $\quad my'' + by' + ky = 0 \ , \quad m, b, k > 0$

Homog Sol $\quad y_h$

We know that $\lim_{t \to 0} y_h = 0$ in every case,

as long as $m, b, k > 0$.

Particular Sol $\quad y_p(t) = A \cos \gamma t + B \sin \gamma t$

It turns out that $y_p(t) = \dfrac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2 \gamma^2}} \sin(\gamma t + \theta)$

Do it!

4.10.16

$$y(t) = y_h(t) + y_p(t)$$

Since $y_h(t) \to 0$ as $t \to \infty$, $y_h(t)$ is called the transient solution.

$y_p(t)$ is called the steady-state solution since

$y(t) \to y_p(t)$ as $t \to \infty$.

The maximum amplitude of the steady-state solution

is $\begin{cases} \dfrac{F_0}{k} & \text{if } b^2 > 2mk \ \& \ \gamma = 0 \\[2ex] \dfrac{F_0/b}{\sqrt{\dfrac{k}{m} - \dfrac{b^2}{4m^2}}} & \text{if } b^2 < 2mk \ \& \ \gamma = \sqrt{\dfrac{k}{m} - \dfrac{b^2}{2m^2}} \end{cases}$

$b^2 < 2mk \Rightarrow b^2 < 2mk < 4mk \Leftrightarrow$ Under damped System

4.10.17 →

$$y_p(t) = \frac{F_0}{\sqrt{(k-m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta)$$

Angular Resonance Frequency $\gamma = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$,

Natural Resonance Frequency $= \frac{\gamma}{2\pi}$, where

$b^2 < 2mk$.

To find the steady-state response of a mass-spring system with forced vibration and damping we only need to find the particular solution.

—4.10.18

Ex A spring with spring constant 16 N/m is attatched to a mass of 8 kg and allowed to more horizontally with the coefficient of friction 0.2 $\frac{N\text{-}sec}{m}$. If the mass is subjected to an external force of $F(t) = 4\cos 2t$ N, determine the steady state response of the system. What is the resonance frequency if $F(t) = F_0 \cos \gamma t$?

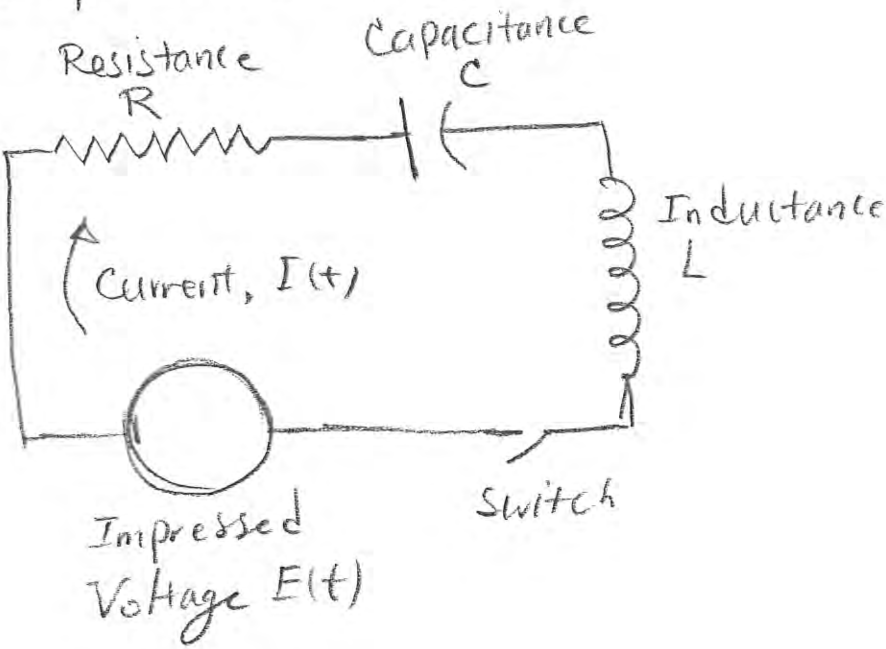
4.10.19 →

← 4.10.20

ENI →

Sections 3.5 & 5.7

Consider a simple series circuit.



Current	$I = I(t)$	Amperes, A
Charge	$Q = Q(t)$	Coulomb, C
Resistance	R	Ohm, Ω
Capacitance	C	Farads, F
Inductance	L	Henrys, H
Voltage	$E = E(t)$	Volts, V

EN 2

The current is the rate of change of charge:

$$I = \frac{dQ}{dt}$$

Voltage drop across the resistor $E_R = IR$

Voltage drop across the capacitor $E_C = \frac{Q}{C}$

Voltage drop across the inductance $E_L = L\frac{dI}{dt}$

$1V = 1\Omega \times 1A$, $1V = 1\ \text{Coulomb}/1\ \text{Farad}$

$1V = 1\ \text{Henry} \times \frac{1\ \text{Amp}}{1\ \text{sec}}$

Kirchoff's 2nd Law In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of circuit

$$E_L + E_R + E_C = E(t)$$

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

EN3 $\rightarrow$

$$L \frac{dI}{dt} + RI + \frac{1}{C} Q = E(t)$$

Since $I = \frac{dQ}{dt}$, $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$, and

$$L \frac{d^2Q}{dt} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$

We could also take the derivative of each side wrt time t.

$$\frac{d}{dt}\left(L \frac{dI}{dt} + RI + \frac{1}{C} Q\right) = \frac{d}{dt} E(t)$$

$$L \frac{d^2I}{dt} + R \frac{dI}{dt} + \frac{1}{C} I = E'(t)$$ since $I = \frac{dQ}{dt}$

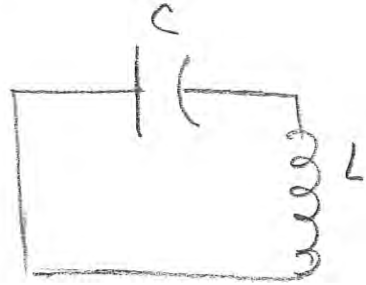
← EN4

The reason for very fast advance in electricital & electronic networks is that mathematically they are identical to spring-mass-dashpot system which has been known much earlier!

Mechanical System	Electrical Circuit
$my'' + by' + ky = F(t)$	$LQ'' + RQ' + \frac{1}{C} Q = E(t)$
Displacement y	Charge Q
Velocity y'	Current $I = Q'$
Mass m	Inductance L
Damping b	Resistance R
Spring Constant k	Elastance $= \frac{1}{\text{Capacitance}}$ $\frac{1}{C}$
Impressed Force $F(t)$	Impressed Voltage $E(t)$

EN5 →

Ex Find the charge in a circuit with a capacitor, with capacitance C, and an inductor, with inductance L.



$$LQ'' + \frac{1}{C}Q = 0$$

$$Q(t) = C_1 \cos\left(\sqrt{\frac{1}{LC}}\,t\right) + C_2 \sin\left(\sqrt{\frac{1}{LC}}\,t\right)$$

Note: The period of this circuit is $\dfrac{2\pi}{\sqrt{\frac{1}{LC}}} = 2\pi\sqrt{LC}$.

The natural frequency of this circuit is

$$\frac{1}{2\pi\sqrt{LC}}$$

← EN 6

Initial Value Problems

$$LQ'' + RQ' + \frac{1}{C} Q = E(t)$$

$Q(0) = Q_0$ initial charge

$Q'(0) = I(0) = I_0$ initial current

$$LI'' + RI' + \frac{1}{C} I = E'(t)$$

$I(0) = I_0$ initial current

$I'(0) = I_1$

$I'(0) = I_1$ is known if we know I_0 & Q_0 since

$$LI' + RI + \frac{1}{C} Q = E(t) \Rightarrow$$

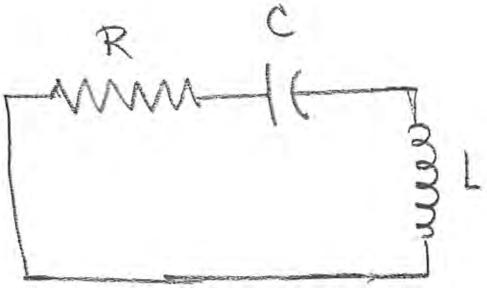
$$LI'(0) + RI(0) + \frac{1}{C} Q(0) = E(0) \Rightarrow$$

$$I'(0) = \frac{1}{L}\left[E(0) - RI_0 + \frac{1}{C} Q_0\right]$$

EN7 →

Ex Find the current in an electrical circuit with no impressed voltage.

$$LI'' + RI' + \frac{1}{C} I = 0$$



(c) Overdamped: $R^2 - \frac{4L}{C} > 0$

$$I(t) = Ae^{r_1 t} + Be^{r_2 t} \quad \text{where}$$

$$r_1, r_2 = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

← EN 8

(ii) Critically Damped $R^2 - \frac{4L}{C} = 0$

$$I(t) = A e^{-\frac{R}{2L}t} + B t e^{-\frac{R}{2L}t}$$

(iii) Underdamped $R^2 - \frac{4L}{C} < 0$

$$I(t) = A e^{\alpha t} \cos\beta t + B e^{\alpha t} \sin\beta t$$ when

$$\alpha = -\frac{R}{2L} \text{ and } \beta = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L}$$

Chapter 6 Theory of Higher-Order Linear Differential Equations

Sec 6.1 Basic Theory of Linear Differential Equations

We want to solve n-th-order linear ODE's

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

with $a_0, \ldots, a_n$ & b_n continuous, real-valued functions on the interval $\alpha < x < \beta$ and $a_n \neq 0$ in this interval. Divide both sides by $a_n(x)$ we get

← 6.1.2

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = g(x)$$

with $P_i(x) = \dfrac{a_{n-i}(x)}{a_n(x)}$, $i = 1, \ldots, n$ & $g(x) = \dfrac{b(x)}{a_n(x)}$.

let $L(y) = \dfrac{d^n y}{dx^n} + P_1(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(x)\dfrac{dy}{dx} + P_n(x)y$

or $L(y) = (D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n)(y)$.

It is easy to show that L is a linear operatior.

That is $L(c_1 y_1 + \cdots + c_n y_2) = c_1 L(y_1) + \cdots + c_n L(y_n)$.

6.153 →

6.1.4

Also, if $y_1, \dots, y_n$ are solutions of the homogeneous equation $L(y)=0$, then their linear combination $y = c_1 y_1 + \dots + c_n y_2$ is also a solution of $L(y)=0$

<u>Def</u> Functions $f_1, \ldots, f_n$ are said to be linearly dependent on the interval I if for n constants $k_1, \ldots, k_n$, not all zero, $k_1 f_1(x) + \cdots + k_n f_n(x) = 0$ for all x in I.

Functions $f_1, \ldots, f_n$ that are not linearly dependent are called linearly independent.

Functions $f_1, \ldots, f_n$ are said to be linearly independent on the interval I if for some constants $k_1, \ldots, k_n$ $k_1 f_1(x) + \cdots + k_n f_n(x) = 0$ for all x in I, then $k_1 = \cdots = k_n = 0$.

6.1.6

Ex Show that the functions $f(x) = 2x-5$, $g(x)=7$ and $h(x) = 6x+1$ are linearly dependent on $(-\infty,\infty)$.

Ex show that $f_1(x) = x$, $f_2(x) = e^x$ and $f_3(x) = \sin x$ are linearly independent on $(-\infty,\infty)$.

Def The Wronskian of functions $y_1(x), y_2(x), \ldots, y_n(x)$ is defined as

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_n^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}.$$

Alternative notations:

$$W[y_1, \ldots, y_n]$$

$$W[y_1, \ldots, y_n](x)$$

Ex Find the Wronskian of functions $f(x) = x$, $g(x) = \cos x$ and $h(x) = \sin x$.

6.1.8

Theorem Suppose $p_1(x), \dots, p_n(x)$ are cont. on the interval (a, b) Let $y_1, \dots, y_n$ be n solutions of the homogeneous equation

$$L(y) = y^{(n)} + p_1 y^{(n-1)} + \dots + p_{n-1} y' + p_n y = 0.$$

for $\alpha < x < \beta$. Then $y_1, \dots, y_n$ are linearly independent on the interval (a, b) if and only if $W(y_1, \dots, y_n)(x) \neq 0$ for at least one x value in (α, β).

Ex Show that the three solutions $y_1(x) = x$, $y_2(x) = \cos x$ and $y_3(x) = \sin x$ of the ODE $xy''' - y'' + xy' - y = 0$ are linearly independent

← 6.1.10

on the interval $(0, \infty)$.

<u>Theorem</u> Suppose $p_1(x), \dots, p_n(x)$ are continuous on the interval (a,b). Let $y_1(x), \dots, y_n(x)$ be n linearly independent solutions of the homogeneous equation

$$L(y) = y^{(n)} + p_1 y^{(n-1)} + \dots + p_{n-1} y' + p_n y = 0.$$

Then, the general sol of $L(y) = 0$ is

$y = c_1 y_1 + \dots + c_n y_n$ for constants $c_1, \dots, c_n$.

6.1.11 →

The linearly independent solutions $y_1, \dots, y_n$ of $L(y) = 0$ are called Fundamental Solutions and the set $\{y_1, \dots, y_n\}$ is called a Fundamental Set of Solutions.

<u>Theorem</u> Suppose $P_1(x), \dots, P_n(x)$ and $g(x)$ are cont. on the interval (a,b). The general solution of the n-th order linear nonhomogeneous eq

$$\underbrace{y^{(n)} + P_1 y^{(n-1)} + \dots + P_{n-1} y' + P_n y}_{L(y)} = g(x)$$

is $y = y_h + y_p$ where y_h is the general solution of the homogeneous eq $L(y) = 0$

← 6.1.12

and y_p is a (particular) solution of the nonhomogeneous equation $L(y) = g(x)$.

Note: The general sol of $L(y)=0$ is $y_h = c_1 y_1 + \cdots + c_n y_n$ where $y_1, \ldots, y_n$ are n linearly independent solutions of $L(y)=0$.

<u>Theorem</u> Suppose $P_1(x), \ldots, P_n(x)$ and $g(x)$ are cont. on the interval (a,b). Then the IVP

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = g(x)$$

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

, with $a < x_0 < b$, has a unique solution.

Sec 6.2 Homogeneous Linear Equations with Constant Coefficients

Consider the homogeneous linear ODE with the constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

where $a_0, \dots, a_n$ are real constants.

Let's look for a solution of the form $y = e^{rx}$

← 6.2.2

The Auxiliary (Characteristic) Equation:

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

Auxiliary (Characteristic) Roots:

$$r_1, r_2, \dots, r_n$$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$L(y) = 0$$

Charac Eq. $a_n r^n + a_n r^{n-1} + \cdots + a_1 r + a_0 = 0$

Charac. Roots $r_1, r_2, \ldots, r_n$

From these roots we obtain the n linearly indep solutions of $L(y) = 0$, as follows.

(a) Real and Unequal Roots

For each characteristic root r which is real and unequal to any othe root we get a solution

$$y = e^{rx}.$$

6.2.4

(b) Complex and Nonrepeating Roots

We know that complex roots occur in conjugate pairs. For each conjugate pair of characteristic root $r = \alpha \pm i\beta$ which is unequal to any other root we get two real-valued solutions

$$y = e^{\alpha x}\cos\beta x \quad \text{and} \quad y = e^{\alpha x}\sin\beta x$$

(c) Repeated (Multiple) Roots

(i) Suppose we have exactly s real equal roots, say r. Then we get s solutions

$$y = e^{rx}, \; y = x e^{rx}, \ldots, \; y = x^{s-1} e^{rx}$$

(ii) Suppose we have exactly s complex conjugate pair $\alpha \pm i\beta$ roots. Then we get 2s solutions

$$y = e^{\alpha x}\cos\beta x,\ y = x e^{\alpha x}\cos\beta x, \ldots, y = x^{s-1} e^{\alpha x}\cos\beta x$$

$$y = e^{\alpha x}\sin\beta x,\ y = x e^{\alpha x}\sin\beta x, \ldots, y = x^{s-1} e^{\alpha x}\sin\beta x$$

<u>Ex</u> Find the general solutions of the following.

1. $y^{iv} - 3y''' + 3y'' - y' = 0$

6.2.6

2. $y^{(5)} + 2y''' + y' = 0$

6.2.7 →

3. $y^{vi} - 2y''' + 2y = 0$

← 6.2.8

6.2.9 →

Ex Find the solution of the initial value problem

$y''' - 3y'' + 4y' - 12y = 0$, $y(0) = 1$, $y'(0) = 1$,

$y''(0) = 26$.

← 6.2.10

Sec 6.3 The Method of Undetermined Coefficients

Your textbook also covers the annihilator method which provides a proof for the choice of particular solutions in the Method of Undetermined Coefficients. Here, we will only discuss the Method of Undetermined Coefficients.

Consider the nonhomogeneous linear ODE with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x).$$

The following table gives the form of a particular solution to this eq.

←6.3.2

The Particular Solution of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

$g(x)$	$y_p(x)$
$P_n(x) = b_n x^n + \dots + b_1 x + b_0$	$x^s(A_n x^n + \dots + A_1 x + A_0)$
$P_n(x)\, e^{rx}$	$x^s(A_n x^n + \dots + A_0)\, e^{rx}$
$P_n(x)\, e^{\alpha x} \begin{cases} \sin\beta x \\ \text{or} \\ \cos\beta x \end{cases}$	$x^s\left[(A_n x^n + \dots + A_0)\, e^{\alpha x}\cos\beta x + (B_n x^n + \dots + B_0)\, e^{\alpha x}\sin\beta x\right]$

where s is the smallest nonnegative integer $(s = 0, 1, \dots, n)$ which will ensure no term in y_p is a solution to the corresponding homogeneous equation. Equivalently, for the above three cases, respectively, s is the number of times 0 is a root of the auxiliary eq., r is a root of the auxiliary eq., and $\alpha + i\beta$ is a root of the auxiliary equation.

<u>Notes</u>: 1. As in 2nd order equations the Method of Undetermined Coefficients in only applicable to equations with constant coefficients and certain inhomogeneity function $g(x)$.

2. If $g(x) = g_1(x) + g_2(x) + \dots + g_k(x)$. Look for a particular solution y_{p_i} for each g_i, $i = 1, \dots, k$.

Then, the particular solution is

$$y_p = y_{p_1} + y_{p_2} + \dots + y_{p_k}$$

<u>Ex</u> Find a particular solution of the following.

6.3.4

1. $y''' + y' = e^x + 4$

2. $y''' - 3y'' + 4y' - 2y = e^x \sin x$ 6.3.5 $\rightarrow$

← 6.3.6

Ex Find the general solution of

$$y^{(5)} - 2y^{(4)} + 2y''' - 2y'' + y' = -e^{-x}$$

6.3.8

6.3.9 →

Ex Find the solution of $y^{iv} + 2y'' + y = 3x + 4$,

$y(0) = y'(0) = 0$ and $y''(0) = y'''(0) = 1$.

← 6.3.10

Sec 6.4 The Method of Variation of Parameters

We want to find a particular solution of

$$\underbrace{y^{(n)} + p_1(x)\cdot y^{(n-1)} + \cdots + p_n(x)\, y}_{L(y)} = g(x)$$

where we cannot apply the method of Undetermined Coefficients; coefficients of the ODE are not constants and/or $g(x)$ is not of the form for the method of Undetermined coefficients. We will extend the technique for the 2nd order equations

← 6.4.2

First, find the solution to the homogeneous equation $L(y)=0$,

$$y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x).$$

Then, the particular sol is

$$y_p(x) = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n$$

where

$$v_k' = \frac{g(x) W_k(x)}{W(x)}, \quad k = 1, \ldots, n$$

where $W(x)$ is the Wronskian of $y_1, \ldots, y_n$ i

$W(x) = W(y_1, \ldots, y_n)(x)$, and W_k is the

determinant obtained from W by

6.4.3 →

replacing the kth column by the column $(0, \ldots, 0, 1)^T$.

$$W = \begin{vmatrix} y_1 & \cdots & y_{k-1} & y_k & y_{k+1} & \cdots & y_n \\ \vdots & & \vdots & \vdots & \vdots & & \\ y_1^{(n-1)} & & y_{k-1}^{(n-1)} & y_k^{(n-1)} & y_{k+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

kth column

$$W_k = \begin{vmatrix} y_1 & \cdots & y_{k-1} & 0 & y_{k+1} & \cdots & y_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ y_1^{(n-1)} & & y_{k-1}^{(n-1)} & 1 & y_{k+1}^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$$

6.4.4

Ex Use the method of Variation of Parameters to find the general solution of $y''' + y' = e^x + 4$.

Homog. Eq. $y''' + y' = 0$

$$y_h(x) =$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} =$$

6.4.5 $\longrightarrow$

$$W_1 = \left|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right.$$

$$W_2 = \left|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right.$$

$$W_3 = \left|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right|$$

6.4.6

$$V_1' = \frac{g(x)W_1}{W}$$

$$V_2' = \frac{g(x)W_2}{W}$$

$$V_3' = \frac{g(x)W_3}{W}$$

6.4.7 $\rightarrow$

Gen sol $y = y_h + y_p$

$y =$

We solved this problem in sec 6.3, using the method of Undetermined Coefficients, and y_p was $y_p(x) = \frac{1}{2} e^{x} + 4x$.

Notice that our formulas here for a 2nd order equations does reduce to the ones we saw in Chapter 4.

← 6.4.8

$$y'' + p_1(x)y' + p_2(x)y = g(x)$$

Homog Eq. $y'' + p_1(x)y' + p_2(x)y = 0$

Lin. Indep. solutions: y_1 & y_2

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} =$ and

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} =$

Now, $y_p = v_1 y_1 + v_2 y_2$ with

$v_1' = \dfrac{g(x) W_1}{W} =$ and $v_2' = \dfrac{g(x) W_2}{W} =$

which are the same formulas as in chapter 4.

6.4.9 →

Ex Find the general solution of $y''' + y' = \sec x$.

Homog Eq.: $y''' + y' = 0$

As before, $y_1 = 1$, $y_2 = \cos x$, $y_3 = \sin x$

$$y_h(x) = C_1 + C_2 \cos x + C_3 \sin x$$

Also, as before

$W = 1$

$W_1 = 1$

$W_2 = -\cos x$

$W_3 = -\sin x$

6.4.10

6.4.11 $\rightarrow$

<u>Ex</u> Find the solution of $y''' + y' = \sec x$,

$y(0) = 2$, $y'(0) = 1$, and $y''(0) = -2$.

From the last problem,

Gen sol, $y(x) = C_1 + C_2 \cos x + C_3 \sin x +$

$\ln|\sec x + \tan x| - x\cos x + \sin x \ln|\cos x|$

6.4.12

6.4.13 →

Ex Find a particular solution of

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \quad x > 0,$$

if $y_1 = x$, $y_2 = x^2$, and $y_3 = \frac{1}{x}$ are three linearly independent solutions of its corresponding homogeneous equation

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 2x$$

So, $g(x) = 2x$

6.4.14

6.4.15→

← 6.4.16

The Method of Reduction of Order

Given one solution y_1 of the homogeneous eq.

$$\underbrace{y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_n(x)\, y}_{L(y)} = 0$$

we can look for a 2nd Lin. Indep Sol of the form $y = v y_1$. By plugging $y = v y_1$ into our nth order homogeneous equation we will get an ODE in terms of $w = v'$ of order $n-1$. If we can solve this reduced equation to get $n-1$ Lin. Indep solutions $w_1, \ldots, w_{n-1}$. Then we

6.4.18

Calculate $v_1, \ldots, v_{n-1}$ where $v_i' = w_i$, $i = 1, \ldots, n-1$

and find n Lin. Indep solutions of $L(y) = 0$,

y_1, $y_2 = v_1 y_1$, $y_3 = v_2 y_1$, $\ldots$, $y_n = v_{n-1} y_1$

However, the reduced $(n-1)$ order equations is usually not significantly easier to solve than the original equation. Thus, in general, the Method of Reduction of Order is seldom useful in higher order equations.

Chapter 7 Laplace Transform

Sec 7.1 Introduction: A mixing Problem

Please read section 7.1. Your books gives an introduction to Laplace transform and a reason for this method.

Sec 7.2 Definition of the Laplace Transform

An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} k(s,t) f(t)\, dt$$

wherein a given func f is transformed into another func F by means of an integral. $F(s)$ is said to be the transform of $f(t)$ and k is called the kernel of

← 7.2.2

the transformation Two integral transforms most often used are the Fourier and Laplace transforms.

<u>Def</u> Laplace Transform - Laplace transform of $f(t), t \geq 0$, denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$ is defined by the eq.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt, \quad t \geq 0,$$

assuming this integral converges.

<u>Question</u>: For what kind of functions f, $F(s)$ exists?

That is, for what kind of functions f,

$$\int_0^{\infty} e^{-st} f(t)\, dt = \lim_{\ell \to \infty} \int_0^{\ell} e^{-st} f(t)\, dt \text{ exists?}$$

Your book uses N in place of ℓ. Your calculus book used t which we cannot use since the dummy variable is t.

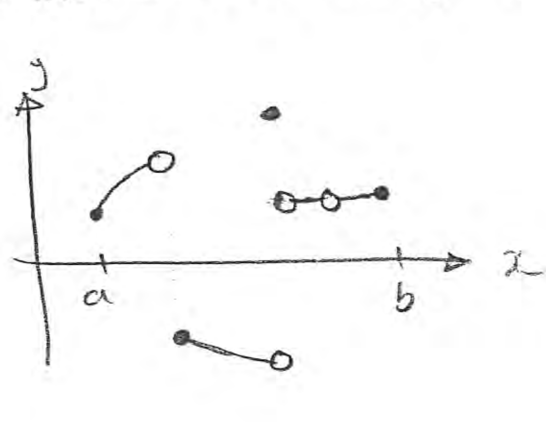
Ex Discuss Laplace transform of $f(t) = e^{at}$, $t \geq 0$, where a is a real nonzero constant.

$$\int_0^{\infty} e^{-st} e^{at}\, dt =$$

7.2.4

So, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, if $s > a$.

<u>Def</u> A func $f(t)$ is said to be piecewise continuous on the interval $[a,b]$ if it is cont. at every value t in $a \le t \le b$, except for a finite number of holes or jump discontinuities.

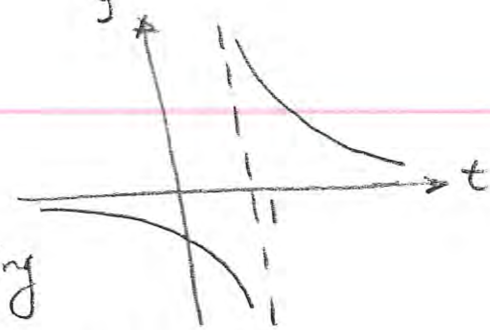


This definition implies that a piecewise cont. fun on $[a,b]$ is also bounded in $[a,b]$.

<u>Def</u> A func f is said to be piecewise continuous on $[0,\infty)$ if $f(t)$ is piecewise cont. on $[0,N]$ for all $N>0$.

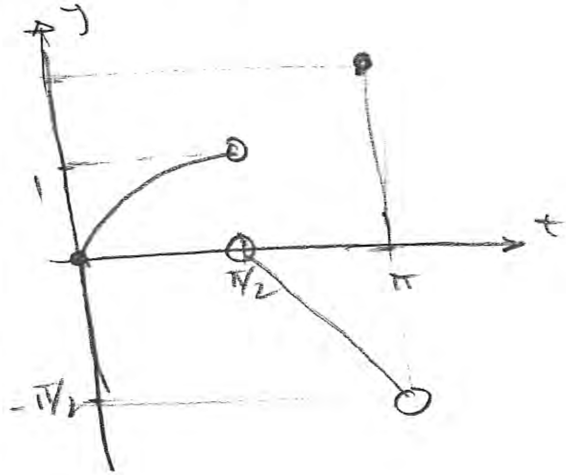
<u>Ex</u> —

1. $f(t) = \dfrac{1}{t-1}$



$f(t)$ is not piecewise cont. on any interval containing $t=1$.

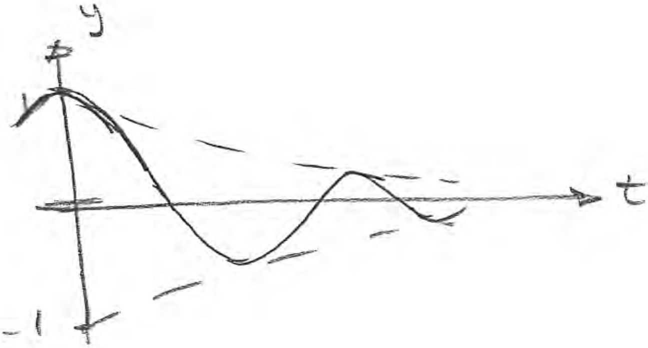
2. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ \frac{\pi}{2} - t, & \frac{\pi}{2} < t < \pi \\ 2, & t = \pi \end{cases}$



is piecewise cont. on $[0,\pi]$

← 7.2.6

3. $f(t) = e^{-t}\cos t$ is piecewise cont. on $[0, \infty)$.



<u>Theorem</u> Suppose that

1. f is piecewise cont. on the interval $[0,\infty)$.
2. f is of exponential order α, that is, $|f(t)| \le M e^{\alpha t}$ for $t \geq T$ where T, M and α are constants and $T > 0$, $M > 0$.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ exists for all $s > \alpha$.

<u>Ex</u> Show the Laplace transform of the given func exists and find it.

1. $f(t) = e^{at}$, $t \geq 0$, where a is a constant.

$f(t) = e^{at}$ is piecewise cont. on $[0, \infty)$.

$|f(t)| = |e^{at}| = e^{at} \leq 1\, e^{at}$ for $t \geq 0$. That is, f is of exponential order a. Thus, Laplace transform of f exists for $s > a$. From before,

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{for } s > a.$$

2. $f(t) = \cos bt$, $t \geq 0$.

$f(t) = \cos bt$ is piecewise cont. on $[0, \infty)$

7.2.8

$|f(t)| = |\cos bt| \le 1 \le 1e^{0t}$ for $t \geq 0$. That is, f is of exponential order zero. So, the Laplace transform of f exists for $s > 0$.

$$\mathcal{L}\{\cos bt\} = \int_0^\infty e^{-st} \cos bt \, dt = \lim_{l \to \infty} \int_0^l e^{-st} \cos bt \, dt$$

7.2.9 →

7.2.10

Ex Find the Laplace transform of $f(t) = \begin{cases} 1, & 0 \le t \le \pi/2 \\ 0, & t > \pi/2 \end{cases}$

7.2.11 →

Ex Find the Laplace transform of $f(t) = \sin bt,\ t \geq 0$

We know that

$$\int e^{-st} \sin bt \, dt = \frac{1}{s^2+b^2} e^{-st}(-b\cos bt + s\sin bt)$$

←7.2.12

7.2.13 →

<u>Theorem</u> Laplace Transform is a linear operator. If f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$ respectively, then for $s > \max\{a_1, a_2\}$ the Laplace transform of $c_1 f_1 + c_2 f_2$ exists and

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

<u>Proof</u> Let $s > \max\{a_1, a_2\}$.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} =$$

7.2.14

7.2.15 →

Ex Find the Laplace transform of $f(t) = 2e^{3t} - \cos 5t$.

7.2.16

Here is a brief table of Laplace transforms.

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$

Sec 7.3 Properties of the Laplace Transform

Theorem If f is piecewise continuous on $[0,\infty)$ and is of exponential order α, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \text{ for } s > \alpha + a$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Proof Let $s > \alpha + a$.

7.3.2

Ex Find $\mathcal{L}\{e^{at}\cos bt\}$.

$f(t) = \cos bt$ is of exponential order zero.

$\mathcal{L}\{\cos bt\} = F(s) = \dfrac{s}{s^2+b^2}$ for $s>0$.

<u>Theorem</u> Suppose f is continuous and f' is piecewise continuous on $[0,\infty)$. If f & f' are of exponential order α, then $\mathcal{L}\{f'(t)\}$ exists and
$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ for $s > \alpha$.

<u>Proof for $f'(t)$ continuous.</u> Suppose $s > \alpha$.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$
$$= \lim_{\ell\to\infty} \int_0^\ell e^{-st} f'(t)\,dt$$

7.3.4

7.3.5 →

<u>Ex</u> Given $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$ for $s>0$. Find $\mathcal{L}\{\sin bt\}$.

← 7.3.6

Replacing f' with f in the formula

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

we get

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

<u>Theorem</u> Suppose $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $[0,\infty)$. If $f, f', \ldots, f^{(n)}$ are of exponential order α, then $\mathcal{L}\{f^{(n)}(t)\}$ exists and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)$$

for $s > \alpha$.

Theorem If f is piecewise continuous on $[0,\infty)$ and is of exponential order α, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n} \text{ for } s>\alpha, \text{ where}$$

$$F(s) = \mathcal{L}\{f(t)\}.$$

Ex Find $\mathcal{L}\{t e^{at}\}$.

$f(t) = e^{at}$ is piecewise cont. on $[0,\infty)$ and is of exponential order a.

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{s-a} \text{ for } s>a.$$

For $s>a$,

← 7.3.8

Properties of Laplace Transform

TABLE 7.2	Properties of Laplace Transforms
$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$.	
$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c .	
$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$.	
$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$.	
$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$.	
$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.	
$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$.	

Sec 7.4 Inverse Laplace Transform

Here is an example of using Laplace transform to solve an IVP.

Ex Solve $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = 1$, using Laplace transform.

Let $Y(s) = \mathcal{L}\{y(t)\}$

7.4.2

So, the key to solving initial value problems with Laplace transform is to find the inverse Laplace transform.

<u>Def</u> Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies $\mathcal{L}\{f(t)\}=F(s)$, then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and we write $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

We don't want to use the formula for the inverse Laplace transform of F(s) since

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{-st} F(s)\,ds$$

We will use the Laplace transform table to find the inverse Laplace transform. To do so, we will try to rewrite F(s) in terms of simpler terms that we have in the Laplace transform table and apply the properties of inverse Laplace transform.

The common technique of writing F(s) in simpler forms is partial fractions.

7.4.4

<u>Theorem</u> Inverse Laplace transform is a linear operator –
If $\mathcal{L}^{-1}\{F_1(s)\}$ and $\mathcal{L}^{-1}\{F_2(s)\}$ exist and are continuous on $[0,\infty)$ and C_1 and C_2 are constants,

then $\mathcal{L}^{-1}\{C_1F_1(s)+C_2F_2(s)\}=C_1\mathcal{L}^{-1}\{F_1(s)\}+C_2\mathcal{L}^{-1}\{F_2(s)\}$

<u>Ex</u> Find the inverse Laplace transform of $F(s)=\frac{s}{s^2-s-6}$.

7.4.5 —

7.4.6

Brief Table of Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}, \ s > 0$
e^{at}	$\frac{1}{s-a}, \ s > a$
$t^n, \ n = 1, 2, \cdots$	$\frac{n!}{s^{n+1}}, \ s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \ s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \ s > 0$
$e^{at} f(t)$	$F(s-a)$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \ s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \ s > a$
$t^n e^{at}, \ n = 1, 2, \cdots$	$\frac{n!}{(s-a)^{n+1}}, \ s > a$
$f(at), \ a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
$u(t-a), \ a \geq 0$	$\frac{e^{-as}}{s}, \ s > 0$
$u(t-a) f(t-a), \ a \geq 0$	$e^{-as} F(s)$
$u(t-a) g(t), \ a \geq 0$	$e^{-as} \mathscr{L}\{g(t+a)\}(s)$
$\delta(t-a), \ a \geq 0$	e^{-as}
$\delta(t-a) f(t), \ a \geq 0$	$e^{-as} f(a)$
$c f(t)$	$c F(s)$
$f(t) + g(t)$	$F(s) + G(s)$, where $G(s) = \mathscr{L}\{g(t)\}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Ex Find the inverse Laplace transform of the following.

1. $F(s) = \dfrac{2s^2 - s + 1}{s^3 - s^2 + s - 1}$

2. $Y(s) = \dfrac{s^2}{s^4 - 1}$

3. $F(s) = \dfrac{s+2}{s^2+1}$

4. $Y(s) = \dfrac{2s+3}{s^2+2s+5}$

5. $G(s) = \dfrac{s+4}{s^2+2s+2}$

6. $H(s) = \dfrac{s}{(s^2+1)(s^2+2s+2)}$

7.4.8

7.4.9 →

$\leftarrow 7.410$

7.4.11 →

← 7.4.12

7.4.13 →

←7.4.14

Sec 7.5 Solving Initial Value Problems

The Method of Laplace Transform

$$\begin{tikzcd}
\text{Initial Value Problem} \arrow[d, "\mathcal{L}"] & \text{Solution of the IVP} \\
\text{Algebraic Equation} \arrow[r, "\text{Algebra}"] & \begin{array}{c}\text{Laplace Transform}\\ \text{of the Solution}\end{array} \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

Ex Solve $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = 1$.

← 7.5.2

7.5.3 →

Ex Solve $y''+2y'+2y=\cos t$, $y(0)=1$, $y'(0)=1$.

$$\mathcal{L}\{y''+2y'+2y\}=\mathcal{L}\{\cos t\}$$

$$\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{\cos t\}$$

7.5.4

7.5.5 →

7.5.6

What if the initial condition is not at zero?

Ex Solve $z''+2z'+2z=\cos t$, $z(2\pi)=1$, $z'(2\pi)=1$.

Let $y(t)=z(t+2\pi)$. Then

$$y'(t)=\big(z'(t+2\pi)\big)(1)=z'(t+2\pi)$$

$$y''(t)=\big(z''(t+2\pi)\big)(1)=z''(t+2\pi)$$

$$y(0)=z(0+2\pi)=z(2\pi)=1$$

$$y'(0)=z'(0+2\pi)=z'(2\pi)=1$$

$$z''(t)+2z'(t)+2z(t)=\cos t$$

$$z''(t+2\pi)+2z'(t+2\pi)+2z(t+2\pi)=\cos(t+2\pi)$$

$$y''(t)+2y'(t)+2y(t)=\cos t$$
$$y(0)=1,\ y'(0)=1$$

7.5.8

$$z(t+2\pi) = y(t) \xRightarrow{t \to t-2\pi} z(t) = y(t-2\pi)$$

$z(t) =$

In general, we can solve the IVP

$$ay''+by'+cy=f(t)$$
$$y(0)=y_0\,,\quad y'(0)=y_1$$

by Laplace transform as follows.

Take the Laplace transform of both sides.

$$\mathcal{L}\{ay''+by'+cy\}=\mathcal{L}\{f(t)\}$$
$$a\mathcal{L}\{y''\}+b\mathcal{L}\{y'\}+c\mathcal{L}\{y\}=\mathcal{L}\{f(t)\}$$
$$a(s^2Y-sy(0)-y'(0))+b(sY-y(0))+cY=F$$

where $Y=Y(s)=\mathcal{L}\{f(t)\}$ and $F=F(s)=\mathcal{L}\{f(t)\}$

$$(as^2+bs+c)Y-(as+b)y_0-ay_1=F(s)$$

← 7.5.10

$$Y(s) = \frac{(as+b)y_0 + ay_1}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$$

Then look for a continuous function $y = y(t)$ whose Laplace transform is $Y(s)$.

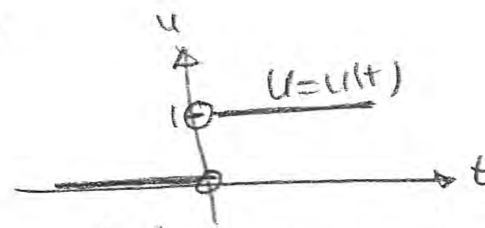
Similarly, we can apply the same method to 1st order and higher order differential equations.

Sec 7.6 Transform of Discontinuous and Periodic Functions

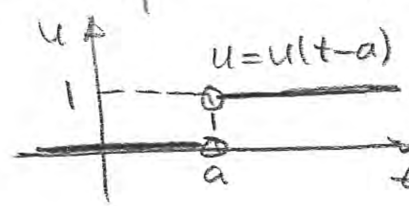
Here we will only cover discontinuous functions.

Def The unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t-a) = \begin{cases} 0, & t-a < 0 \Leftrightarrow t < a \\ 1, & t-a > 0 \Leftrightarrow t > a \end{cases}$$



We can write any piecewise defined function in terms of unit step functions

7.6.2

Ex Write the following in terms of unit step functions

1. $f(t) = \begin{cases} 1, & 0 \leq t < \pi/2 \\ 0, & t \geq \pi/2 \end{cases}$

2. $g(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$

3. $h(t) = \begin{cases} 1, & t < 2 \\ t, & 2 \leq t < 5 \\ t^2, & 5 \leq t < 9 \\ 0, & t > 9 \end{cases}$

7.6.3 →

← 7.6.4

<u>Theorem</u> If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > \alpha \geq 0$, and $a > 0$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a)$$

Moreover, for $f(t) = 1$, $\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \frac{1}{s}$ and for $s > 0$, $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$ & $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a)$.

In addition, $\mathcal{L}\{u(t-a)g(t)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

<u>Proof</u>

7.6.5 →

← 7.6.6

Ex Find the Laplace transform of the following:

1. $f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & t \geq \pi/2 \end{cases}$ 2. $g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \geq \pi \end{cases}$

3. $h(t) = e^{-3t} \mathcal{U}(t-2)$ 4. $k(t) = \mathcal{U}(t-1)\, t^2$

7.6.7 →

7.6.8

7.6.9 →

Ex Find the inverse Laplace transform of the following.

1. $G(s) = \dfrac{e^{-\frac{\pi}{2}s}}{s}$ 2. $H(s) = e^{-\pi s}\dfrac{s}{s^2+1}$

$\leftarrow$ 7.6.10

<u>Ex</u> Solve $y''+y=f(t)=\begin{cases}1, & 0\le t<\pi/2\\ 0, & t\geq \pi/2\end{cases}$, $y(0)=1$, $y'(0)=1$.

7.6.11 →

7.6.12

7.6.13 →

<u>Ex</u> Solve $y'' + 2y' + 2y = g(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$

$y(0) = y'(0) = 0$

←7.6.14

7.6.15 →

7.6.16

Ex Solve $y'' + 3y' + 2y = e^{-3t} u(t-2)$, $y(0) = 2$, $y'(0) = -3$.

7.6.17→

.6.18

Your book also covers rectangular window function

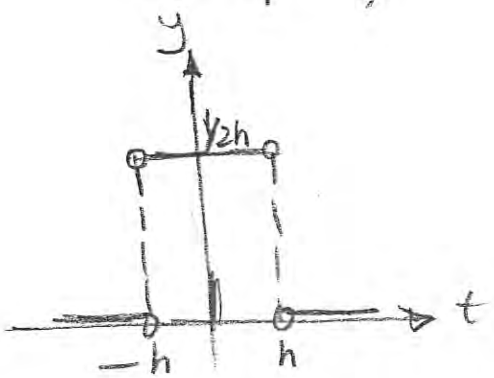
$$\Pi_{a,b}(t) = \begin{cases} 0, & t<a \\ 1, & a<t<b \\ 0, & t>b \end{cases}.$$

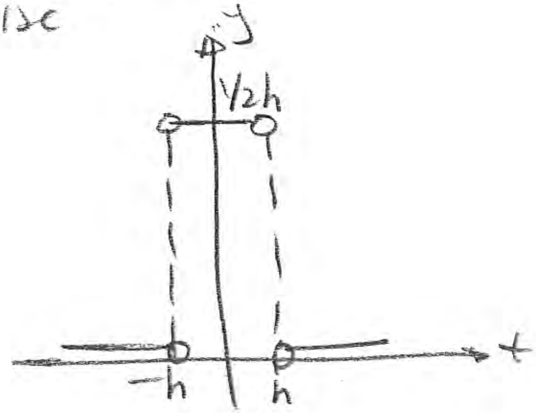
Since $\Pi_{a,b}(t) = u(t-a) - u(t-b)$, it is not necessary to cover it as a separate function!

Sec 7.8 Impulses and the Dirac Delta Function

We want to solve $ay'' + by' + cy = g(t)$, where $g(t)$ is of impulsive nature. For example, impressed voltage or external force of large magnitude that acts over a very short time interval.

Let $d_h(t) = \begin{cases} \frac{1}{2h}, & -h < t < h \\ 0, & \text{otherwise} \end{cases}, \quad h > 0$





$\leftarrow$ 7.8.2

Let $I(h) = \int_{-\infty}^{\infty} d_h(t)\, dt$

$I(h) = \int_{-h}^{h} \frac{1}{2h}\, dt = \frac{1}{2h} t \Big|_{-h}^{h} = \frac{1}{2h}(h-(-h)) = 1$ for all $h > 0$.

Notice that $\lim_{h \to 0} d_h(t) = 0$ for all $t \neq 0$ while

$I(h) = \int_{-\infty}^{\infty} d_h(t)\, dt = 1$ for all $h > 0$.

We define the (generalized) func $\delta(t)$, called the Dirac delta function (or the unit impulse func) via

$$\begin{cases} \delta(t) = 0, & t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \end{cases}$$

and think of it as the limiting behavior of $d_h(t)$ as $h \to 0$.

Shifting from t to $t-a$, we get

$$d_h(t-a) = \begin{cases} \frac{1}{2h}, & -h < t-a < h \Leftrightarrow a-h < t < a+h \\ 0, & \text{Otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} d_h(t-a)\,dt = 1 \quad \text{for all } h > 0.$$

$$\lim_{h \to 0} d_h(t-a) = 0 \quad \text{for all } t \neq a.$$

Dirac delta func (unit impulse func at a):

$$\delta(t-a) = 0, \quad t \neq a$$

$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$$

Suppose $f(t)$ is a func continuous on an interval containing a. Then

7.8.4

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = \lim_{h\to 0} \int_{-\infty}^{\infty} d_h(t) f(t)\,dt$$

$$= \lim_{h\to 0} \int_{a-h}^{a+h} \frac{1}{2h} f(t)\,dt \overset{\text{Mean Value Thm}}{=} \lim_{h\to 0} \frac{1}{2h} f(c)\big((a+h)-(a-h)\big)$$

where $a-h<c<a+h$

$$= \lim_{h\to 0} f(c) = f(a) \quad \text{since } a-h<c<a+h$$

For $a>0$, $\mathcal{L}\{\delta(t-a)\} = \lim_{h\to 0} \int_0^{\infty} e^{-st} d_h(t-a)\,dt$

$$= \lim_{h\to 0} \int_{a-h}^{a+h} e^{-st} \frac{1}{2h}\,dt = \lim_{h\to 0} \frac{1}{2h}\left(-\frac{1}{s} e^{-st}\right)\Big|_{a-h}^{a+h}$$

$$= \lim_{h\to 0} \frac{1}{2h}\left(-\frac{1}{s}\right)\left(e^{-as-sh} - e^{-as+sh}\right) = -\frac{1}{s} \lim_{h\to 0} \frac{e^{-as}\left(e^{-sh}-e^{sh}\right)}{2h}$$

$$= -\frac{e^{-as}}{s} \lim_{h\to 0} \frac{e^{-sh}-e^{sh}}{2h} = -\frac{e^{-as}}{s} \lim_{h\to 0} \frac{-se^{-sh}-se^{sh}}{2}$$

$$= -\frac{e^{-as}}{s}\,\frac{-s-s}{2} = e^{-as}$$

Finally, $\int_{-\infty}^{t} \delta(x-a)\,dx = \lim_{h\to 0} \int_{-\infty}^{t} d_h(x-a)\,dx$

$$= \begin{cases} 0 \,, & \text{if } t<a \text{ since } d_h(x-a)=0 \text{ for } x \le t < a \\ \lim_{h\to 0} \int_{a-h}^{a+h} \frac{1}{2h}\,dx = 1 \,, & \text{if } t>a \end{cases}$$

That is, $\int_{-\infty}^{t} \delta(x-a)\,dx = \begin{cases} 0 \,, & \text{if } t<a \\ 1 \,, & \text{if } t>a \end{cases} = u(t-a)$

So, $\frac{d}{dt} u(t-a) = \delta(t-a)$.

<u>Dirac Delta Func & its properties</u>

$\delta(t-a) = 0$ for $t \neq a$

$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$

$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$ where f is cont. on an interval containing a

7.8.6

$\mathcal{L}\{\delta(t-a)\} = e^{-as}$ for $a > 0$

$\mathcal{L}\{\delta(t)\} = 1$ $(a = 0)$

$\frac{d}{dt} u(t-a) = \delta(t-a)$

Ex Solve $y'' + 2y' + 2y = \delta(t-\pi)$, $y(0)=1$, $y'(0)=0$.

7.8.7—

7.8.8

7.8.9 —

<u>Ex</u> Solve $y''+2y'+2y = \cos t + \delta(t-\pi/2)$, $y(0)=y'(0)=0$

←7.8.10

7.8.11 —

7.8.12

Chapter 8 Series Solutions of Differential Equations

Sec 8.1 Introduction: The Taylor Polynomial Approximation

Power Series $\sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$

We can use a power series to approximate an infinitely many times differentiable function $f(x)$ near $x = x_0$.

Suppose $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$

for x's near x_0.

$\leftarrow$ 8.1.2

$f(x_0) =$

$f'(x) =$

$f'(x_0) =$

$f''(x) =$

$f''(x_0) =$

$f'''(x) =$

$f'''(x_0) =$

$$\left.\begin{aligned} a_0 &= f(x_0) \\ a_1 &= f'(x_0) \\ a_2 &= \frac{f''(x_0)}{2} \\ a_3 &= \frac{f'''(x_0)}{6} \\ &\vdots \end{aligned}\right\} \Rightarrow a_n = \qquad , n = 0, 1, 2 \ldots$$

The Taylor Series expansion of $f(x)$ at x_0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

The nth Taylor polynomial of $f(x)$ at x_0 is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Ex Find the Taylor series expansion of $f(x) = e^x$ at $x = 0$.

8.1.4

8.1.5 →

Ex Find the fourth degree Taylor polynomial of $f(x) = \ln x$ at $x = 1$.

8.1.6

Ex Find the 4th degree Taylor polynomial of the solution of $y''+xy'-y=0$, $y(0)=2$, $y'(0)=-1$.

Let $y=y(x)$ be the solution.

$$P_4(x)=y(0)+y'(0)x+\frac{y''(0)}{2}x^2+\frac{y'''(0)}{6}x^3+\frac{y^{(4)}(0)}{24}x^4$$

↓

8.1.7 →

Ex Find the third degree Taylor polynomial of the solution of $y''+y'+2y=3\ln x$, $y(1)=1$, $y'(1)=1$.

Let $y=y(x)$ be the solution.

$$P_3(x)=y(1)+y'(1)(x-1)+\frac{y''(1)}{2}(x-1)^2+\frac{y'''(1)}{6}(x-1)^3$$

← 8.1.8

Sec 8.2 Power Series & Analytic Functions

Review of Series

$$\sum_{k=1}^{\infty} a_k$$

Series $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right)$ exists and in that case $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$.

The series $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges.

The series $\sum_{k=1}^{\infty} a_k$ converges conditionally if $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} |a_k|$ diverges.

8.2.2

Convergence Tests

Divergence (nth term) Test

Integral Test

Comparison Test

Alternating Series Test

Ratio & Root Tests

Shifting the Summation Index

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{n!} = \sum_{n=0}^{\infty}$$

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{n!} = \sum_{n=}^{\infty} \frac{2^{n}}{}$$

Geometric Series

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \cdots$$

$$\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \frac{a}{1-r}, & \text{if } |r|<1 \\ \text{Diverges}, & \text{if } |r| \geq 1 \end{cases}$$

For $-1 < x < 1$, $\displaystyle\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} \underset{\uparrow}{=} \frac{1}{1-x}$

$a=1, r=x$.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for } -1 < x < 1.$$

Power Series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + \cdots$$

8.2.4

For the power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ exactly one of the following three cases will hold and for each case a radius of convergence ρ is defined.

1. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges only for $x = x_0$ and $\rho = 0$.
2. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges for all x and $\rho = \infty$.
3. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges if $|x-x_0| < R$ and diverges if $|x-x_0| > R$, for some positive number R, and $\rho = R$.

The radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ can be found as follows. Let $a_n = c_n(x-x_0)^n$ and consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

1. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ holds only for $x = x_0$, then $\rho = 0$ and the interval of convergence is $\{x_0\}$.
2. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ holds for all x, then $\rho = \infty$ and the interval of convergence is $(-\infty, \infty)$.
3. $\rho > 0$ is the radius of convergence if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ holds for $|x-x_0| < \rho$ and the interval of convergence is $(x_0-\rho, x_0+\rho)$ plus none, one or both endpoints $x = x_0 \pm \rho$ which must be checked individually.

Equivalently, the radius of convergence can be found as follows. If, $c_n \neq 0$ for large n, and $\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = L$, where $0 \le L \le \infty$, then $\rho = L$. In this version, for L positive and finite, the interval of convergence is $(x_0 - L, x_0 + L)$ plus none, one or both endpoints $x = x_0 \pm L$ which must be checked individually.

Ex Find the radius and interval convergence for

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$$

8.2.6

Properties of Power Series

Suppose both $f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$ and $g(x)=\sum_{n=0}^{\infty} b_n(x-x_0)^n$

are convergent on the open interval $|x-x_0|<\rho$

$(x_0-\rho<x<x_0+\rho)$.

1. $f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n\pm b_n)(x-x_0)^n$ for $|x-x_0|<\rho$

2. If $f(x)=g(x)$, then $a_n=b_n$ for $n=0,1,\dots$

   Or, if $f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n=0$, then $a_0=a_1=\dots=0$.

3. $f'(x)=\sum_{n=1}^{\infty} n\,a_n(x-x_0)^{n-1}$ for $|x-x_0|<\rho$

4. $\int f(x)\,dx=c+\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1}$ for $|x-x_0|<\rho$

← 8.2.8

Note: 1. We can perform the product $f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right]$ by term-by-term multiplication

2. We can perform the division $\frac{f(x)}{g(x)}$ by doing a long division.

Ex Find a_n's so that

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} n a_{n-1} x^{n-1}.$$

8.2.9 —

8.2.10

<u>Fact</u> If a $f(x)$ is equal to the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ in an open interval about x_0, $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ for $x_0-\rho < x < x_0+\rho$, then this power series is the Taylor series expansion of $f(x)$ at x_0.

<u>Def</u> A func $f(x)$ is to be analytic at x_0 if it is equal to its Taylor series expansion at x_0 in an open interval about x_0.

The following functions are analytic on the given interval.

← 8.2.12

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad , -1 < x < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \qquad , -\infty < x < \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad , -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad , -\infty < x < \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{7!} - \cdots \qquad , -1 < x < 1$$

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots , \quad 0 < x < 2$$

If f & g are analytic at x_0, then $f \pm g$, fg & $\frac{f}{g}$ with $g(x_0) \neq 0$, are also analytic at x_0.

In addition, rational functions are analytic on their domains.

8.2.13 →

If a function is analytic at x_0, then it is infinitely-many times differentiable at x_0. So, if a function is not infinitely times differentiable at x_0, then it is not analytic at x_0. For example,

$f(x) = |x-1|$ is not analytic at $x=1$, and

$f(x) = x^{3/2}$ is not analytic at $x=0$.

8.2.14

Sec 8.3 Power Series Solutions to Linear Differential Equations

Consider $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $$y'' + p(x)y' + q(x)y = 0$$

where $p(x) = \dfrac{Q(x)}{P(x)}$ and $q(x) = \dfrac{R(x)}{P(x)}$.

Def A point x_0 is called an ordinary point of
$$y'' + p(x)y' + q(x)y = 0$$
if $p(x) = \dfrac{Q(x)}{P(x)}$ and $q(x) = \dfrac{R(x)}{P(x)}$ are analytic at x_0.

If x_0 is not an ordinary point, it is called a singular point.

8.3.2

Ex Determine the singular points of the following equation.

1. $(x^2-1)y'' + xy' + \frac{x^2-1}{x+2}y = 0$

2. $xy'' + x^2y' + (\sin x)y = 0$

1. $y'' + \frac{x}{x^2-1}y' + \frac{1}{x+2}y = 0$

Since rational functions are analytic on their domain,

the singular points are

2. $xy'' + x^2 y' + (\sin x) y = 0$

$$y'' + x y' + \frac{\sin x}{x} y = 0$$

~~x is analytic everywhere.~~

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{\sin x}{x} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} \quad \text{for all } x$$

This equation has

8.3.4

If x_0 is an ordinary point of the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + p(x)y' + q(x)y = 0,$$

then p & q are equal to their Taylor series expansion about x_0 for $x_0 - h < x < x_0 + h$ for some $h > 0$. That is, p, q are infinitely many times differentiable in this interval and hence the solution y is also differentiable (keep taking derivative of the ODE).

Hence, the general solution y can be expressed in term of a power series about x_0.

Ex Find the first six nonzero terms of power series of the general solution of

$$(1+x^2)y'' - y' + y = 0$$

about $x=0$

$$y'' - \frac{1}{1+x^2}y' + \frac{1}{1+x^2}y = 0$$

Since $p(x) = -\frac{1}{1+x^2}$ and $q(x) = \frac{1}{1+x^2}$ are analytic at $x=0$, the general solution does have a power series expansion at $x=0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$

8.3.6

$$(1+x^2)\,y'' - y' + y = 0$$

8.3.7

— 8.3.8

Ex Find a series solution in powers of x for the equation $y'' - xy' - y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$y'' - xy' - y = 0$$

8.3.10

8.3.1 $\longmapsto$

← 8.3 | 2

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{a_0}{2^k k!} x^{2k} + \sum_{k=0}^{\infty} \frac{2^k k! a_1}{(2k+1)!} x^{2k+1}$$

$$= a_0 \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}.$$

Let $y_1 = \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$ and $y_2 = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$

Then $y = a_0 y_1 + a_1 y_2$. Choosing the pair of values $a_0 = 1, a_1 = 0$ and $a_0 = 0, a_1 = 1$, y_1 and y_2 are each a solution of this ODE.

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots$$

$$y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{15} + \cdots$$

8.3.14

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} =$$

Thus y_1 & y_2 are linearly independent. Hence, the general solution of this equation is

$$y = a_0 y_1 + a_1 y_2 = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$$

Sec 8.4 Equations with Analytic Coefficients

Theorem If x_0 is an ordinary point of

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

or, $$y'' + p(x)y' + q(x)y = 0,$$

then the general solution of this equation is

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x) \quad \text{where}$$

a_0 and a_1 are arbitrary constants and y_1 & y_2 are two linearly independent series solution of this equation and are analytic at x_0.

The radius of convergence for each of the series solutions y_1 & y_2 is at least as large as the

8.4.2

minimum of radii of convergence for the series for $p(x)$ and $q(x)$. Moreover, the radius of convergence of any series solution is at least as large as the minimum distance of x_0 to singular points (real or complex valued) of the ODE.

Ex Find a lower bound for the radius of convergence of the series solution about x_0

1. $(x^2-1)y'' + xy' + \dfrac{x^2-1}{x+2}\, y = 0, \quad x_0 = 0$

2. $xy'' + x^2 y' + (\sin x) y = 0, \quad x_0 = \pi/2$

3. $(x^2+1)(x-2)y'' + xy' + 4y = 0, \quad x_0 = 1$

1. $y'' + \dfrac{x}{x^2-1} y' + \dfrac{1}{x+2} y = 0$, $x_0 = 0$

Singular points: $x = \pm 1$, $x = -2$

2. $y'' + x y' + \dfrac{\sin x}{x} y = 0$

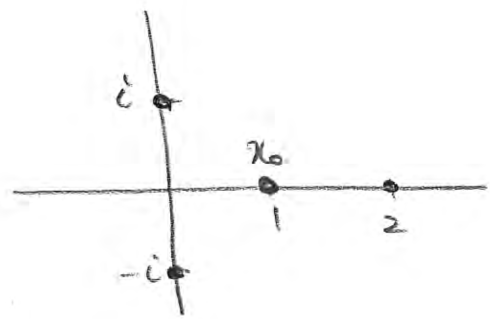
This equation has no singular point.

Radius of convergence of the solution

$\sum_{n=0}^{\infty} a_n (x - \frac{\pi}{2})^n$ is ∞.

8.4.4

3. $y'' + \dfrac{x}{(x^2+1)(x-2)} y' + \dfrac{4}{(x^2+1)(x-2)} y = 0$



The minimum distance of x_0 to singular points is 1. The lower bound for the radius of convergence of the series solution $\sum_{n=0}^{\infty} a_n (x-1)^n$ is 1.

Ex Find the first four nonzero terms of the series expansion of the two linearly independent solution at $x=1$ for $y'' - xy' - y = 0$.

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

8.4.6

Ex Find the first four nonzero terms in a power series expansion of the solution to

$$(x^2+1)y'' - e^x y' + y = 0$$

$$y(0)=1, \quad y'(0)=2$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} +$$

← 8.4.8

← 8.4.8

8.4.10

8.411 →

8.4.12

Ex Find the series solution about $x_0 = 0$ of the general solution of $(1-x)y'' + xy' - y = 0$

8.4.13 →

8.4.14

Sec 8.5 Cauchy-Euler (Equidimensional) Equations

Cauchy-Euler Equation

$$ax^2 y'' + bxy' + cy = 0, \quad a \neq 0,$$

Notice that $x=0$ is a singular point of this equation.
Hence, we cannot find a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$. However, we can solve these equations.

Ex Show that $y = x^r$ is a solution of the Cauchy-Euler equation if $ar^2 + (b-a)r + c = 0$.

8.5.2

The equation

$$ar^2+(b-a)r+c=0,$$

or

$$ar(r-1)+br+c=0$$

is called the characteristic or indicial equation of the Cauchy-Euler equation.

General Solution of $ax^2y''+bxy'+cy=0$

Let r_1 & r_2 be the solutions of the indicial equation $ar^2+(b-a)r+c=0$ (or $ar(r-1)+br+c=0$)

Case (a) r_1 & r_2 are real and unequal.

$$y=c_1|x|^{r_1}+c_2|x|^{r_2}$$

Case (b) $r_1=r_2$

$$y=c_1|x|^{r_1}+c_2|x|^{r_1}\ln|x|$$

Case (c) $r_1, r_2=\alpha\pm i\beta,\ \beta\neq 0$

$$y=c_1|x|^{\alpha}\cos(\beta\ln|x|)+c_2|x|^{\alpha}\sin(\beta\ln|x|)$$

8.5.24

Note: If $x>0$, $|x|=x$, while $|x|=-x$ for $x<0$.

Ex Solve the following

1. $3x^2y''-2xy'-2y=0$, $y(1)=1$, $y'(1)=0$

2. $x^2y''-xy'+y=0$, $x>0$

3. $x^2y''+3xy'+2y=0$, $x>0$

1. $3x^2y''-2xy-2y=0$

$a=3$, $b=-2$, $c=-2$

$ar^2+(b-a)r+c=0$

8.5.5

$\leftarrow$ 8.5.6

2. $x^2 y'' - xy' + y = 0, \ x > 0$

$a = 1, \ b = -1, \ c = 1$

$ar(r-1) + br + c = 0$

3. $x^2 y'' + 3xy' + 2y = 0, \ x > 0$

$a = 1, \ b = 3, \ c = 2$

$ar^2 + (b-a)r + c = 0$

8.5.7 →

Proof of General Solution of $ax^2y''+bxy'+cy=0$

Charac Eq. $ar(r-1)+br+c=ar^2+(b-a)r+c=0$

We know $y=x^r$ is a solution.

Case (a) Characteristic roots r_1 & r_2 are real and unequal.

Case (b) Characteris roots $r_1=r_2$.

$$r=\frac{-(b-a)\pm\sqrt{(b-a)^2-4ac}}{2a}$$

$$r_1=r_2 \Leftrightarrow (b-a)^2-4ac=0 \Leftrightarrow r_1=r_2=-\frac{b-a}{2a}$$

8.5.8

One solution is $y_1 = x^{r_1}$. Look for a 2nd Lin. Indep. solution of the form $y = v y_1 = x^{r_1} v$

8.5.9 →

8.5.10

Case (c) $r_1, r_2 = \alpha \pm i\beta$, $\beta \neq 0$

$y_1 = x^{\alpha - i\beta}$ & $y_2 = x^{\alpha + i\beta}$ are two solutions

8.5.11 →

8.5.12