Calculus II
Fun Problems

A solution must be your own original work. You may discuss your solutions with your instructor.

1. Prove that $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$.

2. Notice that $\left(\sqrt{2}\right)^2 = 2$ and $\left(\sqrt{2}\right)^4 = 4$. Prove analytically that $\left(\sqrt{2}\right)^x \neq x$ for any other real $x$ value.

3. Which one is larger $e^\pi$ or $\pi^e$? You must prove your answer analytically!

4. Suppose $f$ is a 1-1 continuous function with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(x)\,dx$.

5. Suppose $f''$ is a continuous function on $[0,\,1]$ and $f(1) = f'(1) = 0$. Show that $\int_0^1 x^2 f''(x)\,dx = 2\int_0^1 f(x)\,dx$.

6. Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^a} = \frac{\pi}{4}$ for any real number $a$ different from zero.

7. Suppose $f$ is an even continuous function. Show that if $\displaystyle\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx = L$, then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = L$.

8. Suppose $l : \Re^+ \to \Re$ is a function with $l(0) = 0$, $l'(x)$ continuous and $l'(x) \geq 1$ for all $x \geq 0$. Prove that $l$ is an arclength function. That is, $l(x)$ is the arclength of a curve $y = f(t)$ from $t = 0$ to $t = x$.

9. If the wheel whose boundary is the circle $x = \cos t$, $y = 1 + \sin t$, $0 \leq t \leq 2\pi$, is rolled along the $x$-axis, its top touches the line $y = 2$ continuously. Find a wheel whose boundary is not a circle which will do the same. Prove your answer!

10. A ball is released from a height of 10 feet and bounces. Each bounce is $\frac{3}{4}$ of the height of the bounce before. This ball will bounce infinitely many times. Show that this ball will stop bouncing in a finite amount of time, or after about 11 seconds.

11. Show that it is possible to stack identical wooden building blocks in a staircase fashion so that the top block completely extends beyond the endpoint of the bottom block.