Calculus I
Fun Problems

A solution must be your own original work*. You may discuss your solutions with your instructor.

1. If $\lim\limits_{x\to 0^+} f(x) = A$ and $\lim\limits_{x\to 0^-} f(x) = B$, find
   (a) $\lim\limits_{x\to 0^+} f(x^3 - x)$ (b) $\lim\limits_{x\to 0^-} f(x^3 - x)$ (c) $\lim\limits_{x\to 0^+} f(x^2 - x^4)$ (d) $\lim\limits_{x\to 0^-} f(x^2 - x^4)$.
   You must justify your answers!

2. Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{, for } x \neq 0 \\ 0 & \text{, for } x = 0 \end{cases}$. Show that f is continuous and differentiable on the whole real number line, particularly at $x = 0$.

3. Suppose a, b, c, d and e are real numbers. Prove that the five solutions of the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ can not all be real valued if $2a^2 < 5b$.

4. If a and b are positive numbers, prove that the equation
   $$\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x+b} = 0$$
   has two real solutions, one between $\frac{a}{3}$ and $\frac{2a}{3}$ and one between $-\frac{2b}{3}$ and $-\frac{b}{3}$.

5. The area T and an angle γ of a triangle are given. Determine the lengths of the sides a and b so that the side c, opposite the angle γ, is as short as possible.

6. Suppose that $-1 \le ax^2 + bx + c \le 1$ for $-1 \le x \le 1$, where a, b and c are real numbers. Prove that $-4 \le 2ax + b \le 4$ for $-1 \le x \le 1$.

7. Find the smallest value of the positive constant m that will make
   $$mx - 1 + \frac{1}{x}$$
   greater than or equal to zero for all positive values of x.

8. Show that $\displaystyle\int_0^1 x^5\,dx = \lim_{n\to\infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6}$ and then evaluate the given limit.

9. Suppose $f(x)$ has a continuous derivative on the interval $[0, 1]$. Suppose $f(0) = 0$ and $0 \le f'(x) \le 1$. Prove that $\displaystyle\left[\int_0^1 f(x)\,dx\right]^2 \ge \int_0^1 [f(x)]^3\,dx$.

10. Show that for $0 \le \alpha \le \frac{\pi}{2}$, $\displaystyle\int_0^\alpha \sqrt{1+\cos^2\theta}\,d\theta > \sqrt{\alpha^2 + \sin^2\alpha}$.

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