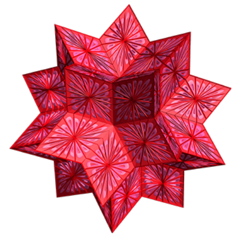
Mathematics Computer Laboratory - Math 1200 - Version 14

Lab 8 - Limits©



Due

You should only turn in exercises in this lab with its title and your name in `Title` and `Subtitle` font, respectively. Edit your document and use appropriate margins to get a polished and neat document.

1. This lab considers limits from several points of view. Then, it considers the special limit that leads to the derivative of a function.

2. Here is an informal review of limits.

   The **right limit** of $f(x)$ as x approaches a:

   $\lim_{x\to a^+} f(x)$ is the value the function $f(x)$ approaches, if any, as x values larger than a get close to a.

   The **left limit** of $f(x)$ as x approaches a:

   $\lim_{x\to a^-} f(x)$ is the value the function $f(x)$ approaches, if any, as x values smaller than a get close to a.

   The **limit** of $f(x)$ as x approaches a:

   $\lim_{x\to a} f(x) = L$ if both the right limit and the left limit values exist and they are equal to L.

   An important fact is that the value of $\lim_{x\to a} f(x)$ is independent of $f(a)$. That is, the limit depends on the values of the function **near** $x = a$ but not $f(a)$ itself. Also, a can be $+\infty$ or $-\infty$ and the value of a limit may also be $+\infty$ or $-\infty$.

   There are three general ways of computing a limit: numerically, graphically and algebraically.

3. Numerically: Use the command `Table` to keep track of numerical approximations to limits.

   (a) `f[x_]:=Sin[x]/Abs[x]` This defines $f(x) = \frac{\sin x}{|x|}$.

   (b) Let's generate numerical approximations to $\lim_{x\to 0^+} \dfrac{\sin x}{|x|}$.

   `Table[{x, NumberForm[N[f[x]],9]}, {x, 0.1, 0.01, -0.01}]//TableForm` This table gives x and its corresponding value of $f(x)$ starting from $x = 0.1$ and ending at $x = 0.01$ in steps of 0.01. Notice that $f(0) = \frac{\sin 0}{|0|}$ is undefined so we stop at 0.01.

   Redo the last table starting from $x = 0.01$ and ending at $x = 0.001$ in steps of 0.001: `{x, 0.01, 0.001, -0.001}`. The values of this function seem to approach 1 as positive x values get close to zero: The prediction is $\lim_{x\to 0^+} \dfrac{\sin x}{|x|} = 1$.

3. Continued.

(c) `Table[{x, NumberForm[N[f[x]],9]}, {x, -0.1, -0.01, 0.01}]//TableForm` This table gives x and its corresponding value of $f(x)$ starting from $x = -0.1$ and ending at $x = -0.01$ in steps of 0.01.

Try `{x, -0.01, -0.001, 0.001}` in the last table. To find values of $f(x)$ to nine decimal places for x's starting from -0.001 and ending at -0.0001 in steps of 0.0001, evaluate the following.

`Table[{x, NumberForm[N[f[x]],9]}, {x, -0.001, -0.0001, 0.0001}]//TableForm`
From these table, it is resonable to conclude that $\lim_{x\to 0^-} \frac{\sin x}{|x|} = -1$. Therefore $\lim_{x\to 0} \frac{\sin x}{|x|}$ does not exist since the right and left limit values are not the same.

(d) To find $\lim_{x\to\infty} \frac{\sin x}{|x|}$, we need to make a table for large values of x.

`Table[{x, N[f[x]]}, {x, 1000, 10000, 1000}]//TableForm` This table gives the values of ordered pairs $(x, f(x))$ starting with $x = 1000$ and ending with $x = 10000$ in steps of 1000. The command `N` is used to obtain the decimal approximation of $f(x)$.

Try `{x, 10000, 100000, 10000}` in the last table. Although the sign of $f(x)$ changes, it appears that $\lim_{x\to\infty} \frac{\sin x}{|x|} = 0$.

(e) Consider the function $g(x) = x^3 + 1$. Let's find the limit value of the slopes of the secant lines through the points $(1, g(1)) = (1, 2)$ and $(t, g(t))$ as t approaches 1. The slope of the secant line is $m = \frac{g(t)-g(1)}{t-1}$. Evaluate each of the following in a different cell.

`g[x_]:=x^3+1`
`Table[{t, (g[t]-g[1])/(t-1)}, {t, 2, 1.1, -0.1}]//TableForm` Since the expression $\frac{g(t)-g(1)}{t-1}$ is undefined at $t = 1$, the table was stopped before reaching this t-value. Try `{t, 1.1, 1.01, -0.01}` and then `{t, 1.01, 1.001, -0.001}` in the last table.

From the table of values, it is reasonable to conclude that $\lim_{t\to 1^+} \frac{g(t)-g(1)}{t-1} = 3$.

`Table[{t, (g[t]-g[1])/(t-1)}, {t, 0, 0.9, 0.1}]//TableForm` Try `{t, 0.9, 0.99, 0.01}` and then `{t, 0.99, 0.999, 0.001}` in the last table.

From the table values, it appears that $\lim_{t\to 1^-} \frac{g(t)-g(1)}{t-1} = 3$. Therefore we predict that $\lim_{t\to 1} \frac{g(t)-g(1)}{t-1} = 3$.

4. Graphically: Use the command `Plot` to graphically approximate values of limits .

(a) `Plot[f[x], {x, -2Pi, 2Pi}]`
Repeat the last graph with `{x, -1, 1}`. Look at the y-coordinate of a point on the graph to the right side of the y-axis. (Right click on the graph and choose Drawing Tools (or Get Coordinates) in the window that opens up. In the new window, click on the symbol that looks like a cross hairs $\boxed{-|-}$ and move the cursor over the graph.) Then look at the y-values of the points on the graph as x values get closer to zero. You should see y values close to 1. Therefore $\lim_{x\to 0^+} f(x) \approx 1$.

(a) Continued.

Repeat reading the y-coordinates of the points on the graph but this time for negative x values (the points on the graph in the left side of the y-axis) as they get close to the zero (move toward the y-axis). You should see y values close to -1. Therefore $\lim_{x\to 0^-} f(x) \approx -1$. And clearly the limit $\lim_{x\to 0} f(x)$ does not exist since there is a jump in the graph at $x = 0$.

(b) To find $\lim_{x\to\infty} f(x)$, graph this function for large values of x.

`Plot[f[x], {x, 10, 100}]`

Repeat the last graph with `{x, 100, 200}` and then `{x, 900, 1000}` . In each graph, read off the largest and smallest values of y-coordinates of the points on the graph as x as gets larger. (Right click on the graph and choose Drawing Tools (or Get Coordinates) in the window that opens up. In the new window, click on the symbol that looks like a cross hairs $\boxed{-|-}$ and move the cursor over the graph.) These values seem to get closer to zero in each successive graph. Therefore $\lim_{x\to\infty} f(x) \approx 0$.

(c) `msl[t_]:=(g[t]-g[1])/(t-1)` The function $msl(t)$ is the slope of the secant line through points $(1, g(1))$ and $(t, g(t))$ on the graph of $y = g(x)$.

(d) `Plot[msl[t], {t, 0, 2}]` Read off the y-coordinates of the points on the graph close to $t = 1$. These values are close to 3 regardless of the points being slightly to the left or to the right of $t = 1$. Therefore $\lim_{t\to 1} msl(t) = \lim_{t\to 1} \dfrac{g(t) - g(1)}{t - 1} \approx 3$.

5. Algebraically: The methods for finding a limit algebraically are discussed in your calculus class and are the only methods which give exact, not approximate, limit values. Mathematica has a `Limit` command which can be used to find certain limits. The usage of this command is as follows.

`Limit[`expression`, x ->` some value or `Infinity` or `-Infinity]`

The command for the right limit and the left limit are as follows.

`Limit[`expression, x `->` some value, `Direction->`“`FromAbove`”`]`

The command `Direction->`“`FromAbove`” designates the **right limit**.

`Limit[`expression, x `->` some value, `Direction->`“`FromBelow`”`]`

The command `Direction->`“`FromBelow`” designates the **left limit**.

(a) `Limit[f[x], x->0, Direction->`“`FromAbove`”`]` This is $\lim_{x\to 0^+} f(x)$ and you should get $+1$.

(b) `Limit[f[x], x->0, Direction->`“`FromBelow`”`]` This is $\lim_{x\to 0^-} f(x)$ and you should get -1.

(c) From parts (a) and (b), it is clear that $\lim_{x\to 0} f(x)$ does not exist. The newer versions of Mathematica will give the **correct** answer to $\lim_{x\to 0} f(x)$.

`Limit[f[x], x->0]` It is best to first investigate existence of the limit of the function by graphing it or considering a table of its values, before applying the `Limit` command.

(d) `Limit[f[x], x->Infinity]` This is $\lim_{x\to\infty} f(x)$ and you should get 0.

(e) `Limit[(g[t]-g[1])/(t-1), t->1, Direction->"FromAbove"]` You should get 3.
`Limit[(g[t]-g[1])/(t-1), t->1, Direction->"FromBelow"]` Again, you should get 3.

Since $\lim_{t\to 1^+} \dfrac{g(t)-g(1)}{t-1} = \lim_{t\to 1^-} \dfrac{g(t)-g(1)}{t-1} = 3$, we conclude $\lim_{t\to 1} \dfrac{g(t)-g(1)}{t-1} = 3$.

`Limit[(g[t]-g[1])/(t-1), t->1]`

6. Recall the limit definition of the derivative of the function $y = f(x)$ at $x = a$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\,, \text{ assuming this limit exists.}$$

(a) Let's find the derivative of $k(x) = x\sin x$ at $x = \pi/3$. Evaluate each of the following in a different cell.

```
k[x_]:=x Sin[x]
Table[{h, (k[Pi/3 + h]-k[Pi/3])/h}, {h, 0.1, 0.01, -0.01}]//TableForm
Table[{h, (k[Pi/3 + h]-k[Pi/3])/h}, {h, -0.1, -0.01, +0.01}]//TableForm
```
These two tables seem to show the left and the right limits exists and they are equal.
`Limit[(k[Pi/3 + h]-k[Pi/3])/h, h->0]`
`N[%]` It appears that $k'(\pi/3) = \frac{1}{6}(3\sqrt{3}+\pi) \approx 1.39$.

(b) We can use the limit definition to find the derivative of $k(x)$ for any x value: "$k'(x)$".
`Limit[(k[x + h]-k[x])/h, h->0]` You should get `x Cos[x] +Sin[x]`.

(c) Mathematica has several commands for finding the derivative. Evaluate each of the following in a different cell.

`k' [Pi/3]` This is the derivative of $k(x) = x\sin x$ at $x = \pi/3$ and is the same value as in part (a).
`k' [x]` This is $k'(x)$ and is the same expression as in part (b).
`D[k[x], x]` This is the derivative of $k(x)$ with respect to x and is the same expression as in the last part.
The command `D` stands for the derivative, `k[x]` the function whose derivative is desired and `x` the variable with respect which the derivative should be taken.

(d) Recall that the slope of the line tangent to a graph at a given point is the value of its derivative at that point. It is also the limit value of the slope of secant lines through the given point as the second point on the graph is moved toward the given point. Therefore $\lim_{t\to 1} \dfrac{g(t)-g(1)}{t-1}$ is the slope of the tangent line to $y = g(x)$ at $x = 1$ and is also equal to $g'(1)$.

`g' [1]` Is this the same value as $\lim_{t\to 1} \dfrac{g(t)-g(1)}{t-1}$ found earlier?

(e) The equation of the tangent line to the curve $y = g(x) = x^3 + 1$ at $x = 1$ is $y - g(1) = g'(1)(x-1)$ which simplifies to $y = g(1) + g'(1)(x-1)$ or $y = 3x - 1$.

The equation of the secant line through points $(1,\, g(1))$ and $(t,\, g(t))$ is $y - g(1) = \frac{g(t)-g(1)}{t-1}\,(x-1)$ which simplifies to $y = g(1) + \frac{g(t)-g(1)}{t-1}\,(x-1)$ or $y = \frac{g(t)-2}{t-1}\,(x-1) + 2$.

(f) To demonstrate the ideas that the derivative gives the slope of the tangent line, which in turn is the limiting value of the slopes of secant lines, do the following animations.

```
tl[x_]:=g[1]+g′[1](x-1)
sl[x_]:=g[1]+((g[t]-g[1])/(t-1))(x-1)
ListAnimate[Table[Plot[{g[x], sl[x], tl[x]}, {x, -0.25, 2.25},
                  PlotRange->{{-0.25, 2.25}, {-0.5, 10}},
                  PlotStyle->{{RGBColor[1, 0, 0]}, {RGBColor[0, 1, 0]},
                              {RGBColor[0, 0, 1]}}],
      {t, 0, 0.9, 0.1}]]
```

Redo the last animation with `{t, 2, 1.1, -0.1}`.

`Clear[f, g, msl, k, tl, sl]`

Perform each of the following exercises. Explain your work by typing in necessary comments in text cells.

A. Investigate the following limits. Decide if a limit has a value by using either numerical (`Table` command) or graphical (`Plot` command) means. If a limit exists, find its value using the `Limit` command. If a limit does not exist, explain why.

(i) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$

(ii) $\displaystyle\lim_{x\to\infty} \frac{-2x^3 + 150x^2 + 6x + 1999}{5x^3 - 2000x^2 + x + 3}$

(iii) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{|x|}$

B. Graphically, explain why $\displaystyle\lim_{x\to 0^+} \sin\frac{\pi}{x}$ does not exist. For $x = \frac{1}{n}$, $\sin\frac{\pi}{x} = \sin(\frac{\pi}{1/n}) = \sin n\pi$. Consider values of $\sin\frac{\pi}{x}$ for $x = \frac{1}{n}$ where $n = 1, 2, 3, \cdots$ as illustrated by

`Table[{1/n, Sin[n Pi]}, {n, 1, 10, 1}]`.

The y values in the table may make one to believe that the limit exists. Why is the limit value not the y value of the points in the table? Explain!

C. Given $f(x) = x\,e^x$ find each of the following using: 1. The limit definition of the derivative and the `Limit` command and 2. The derivative symbol prime " ′ " or the command `D`.

(i) $f'(1)$

(ii) $f'(x)$

D. Find the equation of the tangent line to the graph of $y = x^2 \sin x$ at $x = \frac{\pi}{2}$. You may use a derivative command.