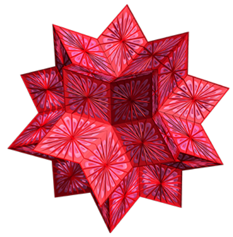
Mathematics Computer Laboratory - Math 1200 - Version 14

Lab 4 - Matrices and Their Applications©



Due

You should only turn in exercises in this lab with its title and your name in `Title` and `Subtitle` font, respectively. Edit your document and use appropriate margins to get a polished and neat document.

1. The objective of this lab is to discuss syntax for matrices, matrix operations and an application of matrices. Open a new Mathematica notebook.

2. Braces, { }, signal a "list" to Mathematica. A list can be used to represent a set, an interval, a table of data, the parameters in a command, a matrix, etc. In $\texttt{Plot}[\mathtt{E}^{\mathtt{x}}, \{\mathtt{x}, -1, 3\}]$, the list $\{\mathtt{x}, -1, 3\}$ indicates that the independent variable is x and the domain is the interval $[-1, 3]$. In $\texttt{Solve}[\{2\mathtt{x} + 3\mathtt{y} == -1, 5\mathtt{x} - 2\mathtt{y} == 7\}, \{\mathtt{x}, \mathtt{y}\}]$, the first list is the system of equations to be solved and the second list indicates the variables this system should be solved for and also the order in which the answer should be given. Now try the following.

   (a) `list1={-3, 2/5, 1.3, Pi, 8, 5}`

   (b) `ListPlot[list1]`
   Notice that the values of list1 are used as the y values and each is matched with a x value according to the order it appear in the list starting from $x = 1$. (Mathematica has probably placed the y-axis at $x = 1$.)

   (c) `ListPlot[list1, AxesOrigin->{0,0}, PlotRange->{{-1,7},{-4,9}}]`
   The command `AxesOrigin->{0,0}` places the origin of the coordinate system at $(0, 0)$ and the command `PlotRange->{{-1,7},{-4,9}}` gives the domain $[-1, 7]$ and the range $[-4, 9]$.

   (d) `list2={{2,-3}, {-1,2/5}, {0,1.3}, {1,Pi}, {4.5,8}, {3,5}}`

   (e) `ListPlot[list2]` This time each element of the list is an ordered pair and graphed as such.

   (f) `ListPlot[list2, Joined->True]`
   The command `Joined->True` connects the points of the graph in order they appear in the list with straight line segments.

3. It is possible to combine lists to generates new ones. In particular, one can input the data for each variable separately, and then combine the two. Let's reproduce list2 by combining list1 with another list.

   (a) `list3={2, -1, 0, 1, 4.5, 3}`

   (b) `list4={list3, list1}`
   Notice that list4 includes both list1 and list3 but appropriate values are not paired up.

   (c) `list5=Transpose[list4]`
   The command `Transpose` matches up corresponding elements of list3 and list1, which makeup list4. (Actually, it is a 6-row, 2-column matrix that we interpret as ordered pairs.)

   (d) `list5==list2` A true output confirms equality of these two lists.

   (e) `Clear[list1, list2, list3, list4, list5]`

4. You can input a matrix as a list by inputting each row as a list in the order they appear in the matrix. You can add or subtract matrices using + and − signs. The symbol "." is the multiplication sign for matrices.

   (a) `m1={{1,2}, {3,4}}` This is the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

   (b) `TableForm[m1]` The matrix $m1$ is listed as in a spreadsheet (table form).

   (c) `MatrixForm[m1]` The matrix $m1$ is displayed in its standard matrix form.

   (d) `m2={{5,6}, {7,8}}` This is the matrix $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$.

   (e) `s=m1+m2` s is the sum of $m1$ and $m2$ which is obtained by adding their corresponding elements.

   (f) `m1-2*m2` This is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 10 & 12 \\ 14 & 16 \end{pmatrix} = \begin{pmatrix} -9 & -10 \\ -11 & -12 \end{pmatrix}$.

   (g) `p=m1.m2` This gives the product of matrices $m1$ and $m2$.

   (h) `MatrixForm[p]`

   (i) `m1inv=Inverse[m1]` The command `Inverse` gives the inverse of the matrix.

   (j) `m1inv.m1` Remember that the product of a matrix and its inverse is the identity matrix.

   (k) `m1.m1inv`

   (l) `tm2=Transpose[m2]` The `Transpose` command interchanges rows and columns of the matrix.

   (m) `MatrixForm[tm2]` Compare this matrix with the $m2$ matrix.

   (n) `Clear[m1, m2, s, p, m1inv, tm2]`

A matrix equation,

$$AX = B\,,$$

where A is the coefficient matrix, X the column matrix of the unknowns and B the column matrix of constants, may be solved in several ways. For example consider the following system of equations and its matrix form.

$$\begin{array}{l} x + 2y - 2z = -1 \\ x + 3y - z = 1 \\ x + 2y - z = 3 \end{array} \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & 2 & -2 \\ 1 & 3 & -1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

1st Solution If the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 3 & -1 \\ 1 & 2 & -1 \end{pmatrix}$ is invertible, then

$$X = A^{-1}B$$

2nd Solution Solve this matrix equation as a system of equations using `Solve` or `NSolve` commands.

5. Let's try these two methods for the above example.

   (a) `mA={{1,2,-2}, {1,3,-1}, {1,2,-1}}; mX={x,y,z}; mB={-1,1,3};`

   (b) Input each `MatrixForm` in a different cell and inspect them to verify correctness of data entry.
   `MatrixForm[mA]`
   `MatrixForm[mX]`
   `MatrixForm[mB]`

5. Continued.

   (c) `mInvA=Inverse[mA]`

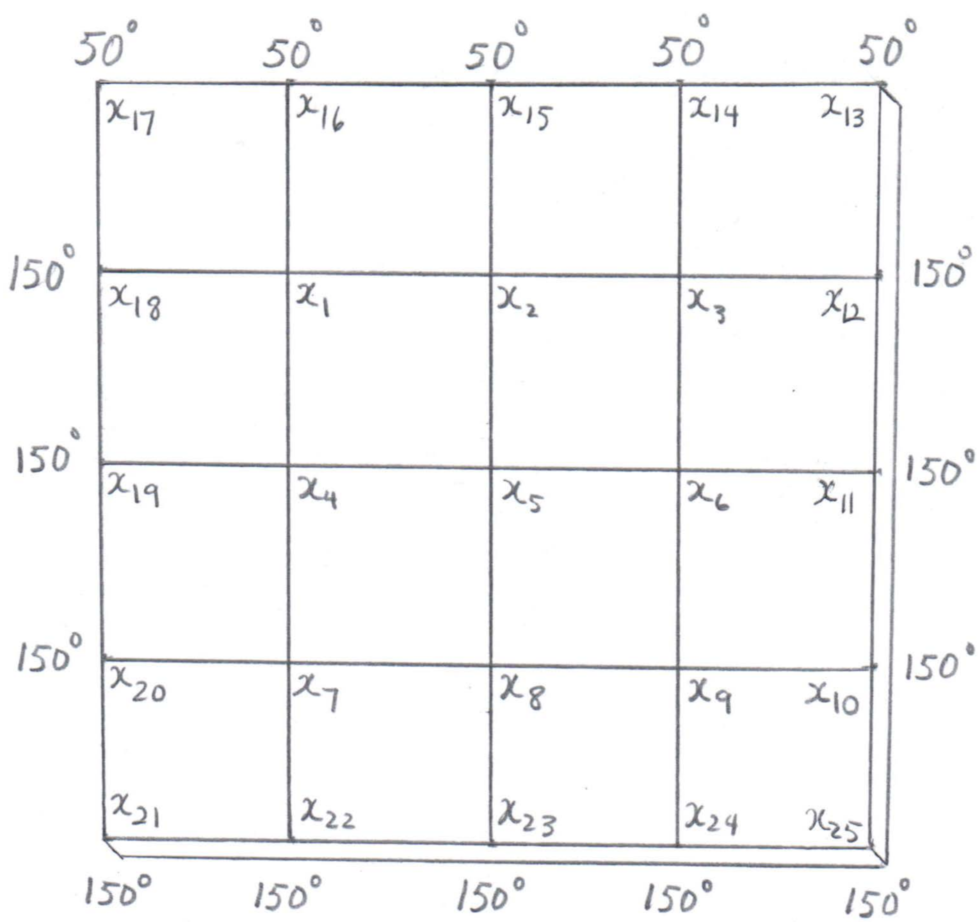
   (d) `mInvA.mB` The x is the first value, y the second and z the last.

   (e) `NSolve[mA.mX==mB, mX]` Did you get the same answer?

   (f) `Clear[mA, mX, mB, mInvA]`

Suppose that an object having some initial internal temperature distribution were subjected to varying amounts of heat at different places of its surfaces. If the temperature of its surfaces are kept fixed, then it is natural to expect that the interior of the object will eventually reach a fixed temperature distribution.

Suppose this object is a square plate with uniform density, negligible thickness and fully insulated lower and upper surfaces. This plate is placed in an environment with fixed temperature of $150\,^oF$ at the left, right and bottom edges and fixed $50\,^oF$ temperature at its top edge. Also, the top-left and top-right vertices have the same $50\,^oF$ temperature as the top edge. See the diagram. We want to find this square's temperature distribution after a long period of time.



Methods for finding the temperature of the points inside this square are beyond the scope of this course. So, to simplify the problem (to discretize it), we divide this square into 16 equal pieces using a 4×4 (4 by 4) grid of uniform mesh size, and try to find the temperature at the 25 vertices of our grid (see the diagram). Notice that we already know the temperature at 16 of these vertices (the one's on the edges of the square).

Each x_i denotes the temperature at the vertex i, $i = 1$ to $i = 25$. Thus $x_{13} = x_{14} = x_{15} = x_{16} = x_{17} = 50$, and $x_{10} = x_{11} = x_{12} = x_{18} = x_{19} = x_{20} = x_{21} = x_{22} = x_{23} = x_{24} = x_{25} = 150$. Now we are left to find the temperature at only nine vertices (the interior vertices). As a further simplification assume that the temperature at any interior vertex is the average of the temperatures of the 8 vertices surrounding (or next to) it. This yields the following system of equations.

$$x_1 = \frac{x_{17} + x_{16} + x_{15} + x_2 + x_5 + x_4 + x_{19} + x_{18}}{8} \qquad x_2 = \frac{x_{16} + x_{15} + x_{14} + x_3 + x_6 + x_5 + x_4 + x_1}{8}$$

$$x_3 = \frac{x_{15} + x_{14} + x_{13} + x_{12} + x_{11} + x_6 + x_5 + x_2}{8} \qquad x_4 = \frac{x_{18} + x_1 + x_2 + x_5 + x_8 + x_7 + x_{20} + x_{19}}{8}$$

$$x_5 = \frac{x_1 + x_2 + x_3 + x_6 + x_9 + x_8 + x_7 + x_4}{8} \qquad x_6 = \frac{x_2 + x_3 + x_{12} + x_{11} + x_{10} + x_9 + x_8 + x_5}{8}$$

$$x_7 = \frac{x_{19} + x_4 + x_5 + x_8 + x_{23} + x_{22} + x_{21} + x_{20}}{8} \qquad x_8 = \frac{x_4 + x_5 + x_6 + x_9 + x_{24} + x_{23} + x_{22} + x_7}{8}$$

$$x_9 = \frac{x_5 + x_6 + x_{11} + x_{10} + x_{25} + x_{24} + x_{23} + x_8}{8}$$

Putting in the known values and simplifying yields the following 9×9 linear system of equations.

$$\begin{array}{rrrrrrrrrcr}
8x_1 & -x_2 & & -x_4 & -x_5 & & & & & = & 450 \\
-x_1 & +8x_2 & -x_3 & -x_4 & -x_5 & -x_6 & & & & = & 150 \\
& -x_2 & +8x_3 & & -x_5 & -x_6 & & & & = & 450 \\
-x_1 & -x_2 & & +8x_4 & -x_5 & & -x_7 & -x_8 & & = & 450 \\
-x_1 & -x_2 & -x_3 & -x_4 & +8x_5 & -x_6 & -x_7 & -x_8 & -x_9 & = & 0 \\
& -x_2 & -x_3 & & -x_5 & +8x_6 & & -x_8 & -x_9 & = & 450 \\
& & & -x_4 & -x_5 & & +8x_7 & -x_8 & & = & 750 \\
& & & -x_4 & -x_5 & -x_6 & -x_7 & +8x_8 & -x_9 & = & 450 \\
& & & & -x_5 & -x_6 & & -x_8 & +8x_9 & = & 750
\end{array}$$

The matrix form of this system is:

$$\begin{bmatrix}
8 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 8 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & -1 & 8 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 8 & -1 & 0 & -1 & -1 & 0 \\
-1 & -1 & -1 & -1 & 8 & -1 & -1 & -1 & -1 \\
0 & -1 & -1 & 0 & -1 & 8 & 0 & -1 & -1 \\
0 & 0 & 0 & -1 & -1 & 0 & 8 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & -1 & 8 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 8
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}
=
\begin{bmatrix} 450 \\ 150 \\ 450 \\ 450 \\ 0 \\ 450 \\ 750 \\ 450 \\ 750 \end{bmatrix}.$$

6. Let's solve this system using Mathematica.

   (a)
```
mA={{8,-1,0,-1,-1,0,0,0,0}, {-1,8,-1,-1,-1,-1,0,0,0},
    {0,-1,8,0,-1,-1,0,0,0}, {-1,-1,0,8,-1,0,-1,-1,0},
    {-1,-1,-1,-1,8,-1,-1,-1,-1}, {0,-1,-1,0,-1,8,0,-1,-1},
    {0,0,0,-1,-1,0,8,-1,0}, {0,0,0,-1,-1,-1,-1,8,-1},
    {0,0,0,0,-1,-1,0,-1,8}}
```

   (b)
```
mB={450, 150, 450, 450, 0, 450, 750, 450, 750}
```

   (c)
```
mX={x1, x2, x3, x4, x5, x6, x7, x8, x9}
```

6. Continued.

   (d) `MatrixForm[mA]` Check for correctness of data entry.
   `MatrixForm[mX]`
   `MatrixForm[mB]`

   (e) `Inverse[mA].mB` Find the decimal approximation (use the `N` command) and record values of x_i's on the diagram in page 3.

   (f) `NSolve[mA.mX==mB, mX]` Compare with answers obtained in the last step.

   (g) `Clear[mA, mB, mX]`

Now do the following exercises in a new Mathematica notebook. Give appropriate comments (type your comments in text cells) to explain your your work.

A. Input the following list of values for x and y, each as a column matrix (7-row, 1-column).

x	1.13	2.56	3.42	4.75	5.39	6.84	7.25
y	3.83	4.15	3.79	1.78	0.23	−0.76	−1.25

   (i) Combine the two lists into a 2-row, 7-column matrix with the x values as the first row.

   (ii) Obtain a matrix of ordered pairs of corresponding x and y values by transposing the matrix obtained in the previous part. Name this matrix `data`.

   (iii) Plot `data`.

   (iv) Plot `data` with its consecutive points connected by straight lines.

   (v) Plot `data` with the origin at $(0,0)$, Domain $= [-1,8]$ and Range $= [-2.5,5]$.

   (vi) Clear the variables.

B. Solve the following system of equations two ways: (i) using inverse matrices and (ii) using `NSolve` command.

$$\begin{array}{rrrrrrcr}
2u & +3v & +4w & -3x & +5y & -6z & = & 8\\
3u & -v & -5w & +x & -3y & +2z & = & -3\\
u & +4v & -2w & +4x & -5y & +3z & = & -1\\
4u & +3v & -5w & +x & -2y & -3z & = & -6\\
3u & -5v & +2w & -3x & -2y & +4z & = & 4\\
3u & -2v & -4w & +2x & -4y & -5z & = & -6
\end{array}$$

C. Repeat the plate temperature application for the temperatures of $60\,^oF$ and $-5\,^oF$ in place of $150\,^oF$ and $50\,^oF$, respectively. Be sure to derive the correct equations for the new temperatures. State the new equations in a text cell. After the mathematical computations, list the answers as $x_1 = \cdots$, $x_2 = \cdots$, etc., in a text cell. Explain your work. You may typeset x_i as xi.