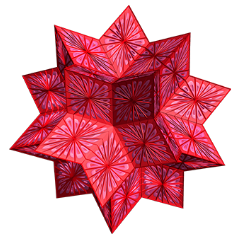
Mathematics Computer Laboratory - Math 1200 - Version 14

Lab 9 - Applications of Derivatives©



Due

You should only turn in exercises in this lab with its title and your name in `Title` and `Subtitle` font, respectively. Edit your document and use appropriate margins to get a polished and neat document.

1. This lab discusses finding derivatives using Mathematica. Then, it considers applications of derivatives in graphing, optimization and as a stand alone mathematical entity.

2. Both the prime symbol " ′ " and the command `D` can be used for finding derivatives. However, the command `D` is more convenient for higher order derivatives. Input and evaluate the following.

   (a) `f[x_]:=x^2 Sin[3x+1]` This defines $f(x) = x^2 \sin(3x+1)$.

   (b) `f'[x]` Check this answer by writing out the derivative of f on your own.

   (c) `D[f[x], x]` You should get the same answer as in the last part.
   The command `D` stands for the derivative, `f[x]` the function whose derivative is desired and `x` the variable with respect which the derivative should be taken.

   (d) `f''[x]` This gives $f''(x)$.

   (e) `D[f[x], {x, 2}]` You should get the same answer as in the last part.
   The list `{x, 2}` indicates the variable with respect which the derivative should be taken and the order of the derivative.

   (f) $f^v(x)$ the fifth derivative of $f(x)$ with respect to x can be evaluated as follows.
   `D[f[x], {x, 5}]`
   `Clear[f]`

3. To find the derivative of a function defined implicitly we must use the command `D`.

   (a) Find y' in terms of x and y if $x^2y^2 + x\sin y = 1$.
   `D[x^2 y[x]^2 + x Sin[y[x]]==1, x]` By `y[x]` we mean that y is a function of x.

   Now solve the last equation for y'.
   `Solve[% , y'[x]]` Find y' on your own and compare with this answer.

4. In the calculus course, the first and second derivatives of a function are used to find such things as: local extreme points, inflection points, where it increases, where it decreases, where it is concave up and where it is concave down. Review these topics from calculus as necessary.

(a) Find the intervals on which the function $f(x) = x^4 - 4x^3 - 48x^2 + 104x - 3$ is increasing or decreasing and its local extreme points .

Input each of the following in a different cell.
`f[x_]:=x^4-4x^3-48x^2+104x-3`
`fp[x_]:=f'[x]` Here `fp[x]`, short for f prime of x, is defined to be $f'(x)$ and is used so that the derivative does not need to be recomputed every time we need it.

You can see the derivative by evaluating the following.
`fp[x]`
`NSolve[fp[x]==0,x]` This gives the x values at which the derivative is zero.

Notice that `fp[x]` is defined everywhere. Therefore the critical points (or numbers) are $x \approx -4.196$, $x = 1$ and $x \approx 6.196$

To find the sign of f', let's look at its graph.
`Plot[fp[x], {x, -6, 8}]` We use a domain large enough to include all x values at which f' is zero or undefined.

Therefore f is decreasing on $(-\infty, -4.196)$ and $(1, 6.196)$ and f is increasing on $(-4.196, 1)$ and $(6.196, \infty)$.
`f[-4.196]`
`f[1]`
`f[6.196]`
Also the points $(-4.196,\ f(-4.196)) = (-4.196,\ -679)$ and $(6.196,\ f(6.196)) = (1,\ -679)$ are local minimum points of f while the point $(1,\ f(1)) = (1,\ 50)$ is the local maximum point.

These results can be seen in the graph of f.
`Plot[f[x], {x, -8, 10}]`

(b) Find the intervals on which the function $g(x) = \frac{x^2+x+1}{x^2+1}$ is concave up or concave down and its inflection points .
`g[x_]:=(x^2+x+1)/(x^2+1)`
`gdp[x_]:=g''[x]` Here `gdp[x]`, short for g double prime of x, is defined to be $g''(x)$.
You can look at the simplified form of the second derivative by evaluating the following.
`Simplify[gdp[x]]`
`NSolve[gdp[x]==0,x]` This gives the x values at which the second derivative is zero.

Notice that `gdp[x]` is defined everywhere. Therefore the inflection points can occur only at x values in the domain of g at which the second derivative is zero; $x \approx -1.73$, $x = 0$ and $x \approx 1.73$.

To find the sign of g'', let's look at its graph.
`Plot[gdp[x], {x, -3, 3}]` We use a domain large enough to include all x values at which g'' is zero or undefined.

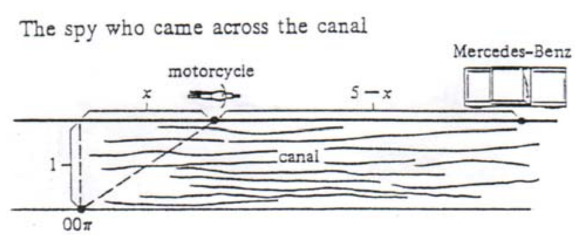
(b) Continued.

Therefore g is concave up on $(-1.73,\, 0)$ and $(1.73,\, \infty)$
and g is concave down on $(-\infty,\, -1.73)$ and $(0,\, 1.73)$.
`g[-1.73]`
`g[0]`
`g[1.73]`
The graph of g changes concavity at $x \approx -1.73$, $x = 0$ and $x \approx 1.73$. Thus the inflection points are $(-1.73,\, g(-1.73)) = (-1.73,\, 0.57)$, $(0,\, g(0)) = (0,\, 1)$ and $(1.73,\, g(1.73)) = (1.73,\, 1.43)$.

It is often not possible to show all relevant features on a general graph. You must be able to calculate the location of features independently. Then you can specify the `PlotRange` in Mahematica to view each feature.
`Plot[g[x], {x, -6, 6}]` Can you see the inflection points?
`Clear[f, fp, g, gdp]`

5. An important application of derivative is in optimization. Here we discuss one problem.

(a) Agent 00π is waiting at the edge of a straight canal 1 mile wide, in a motorboat capable of doing 40 mph . There is a straight road along the opposite edge of the canal; her partner will have a 50 mph motorcycle waiting for her wherever she lands. At midnight she will receive a package to be delivered to a man in a Mercedes-Benz 5 miles down the road on the opposite side of the canal. Mission: Deliver the package in the shortest amount of time. Problem: Where should her partner park the motorcycle?



From the picture, she travels $\sqrt{1+x^2}$ miles over the canal and $5-x$ miles on the road. We want to find the x value at which the total travel time is minimum. The total travel time is

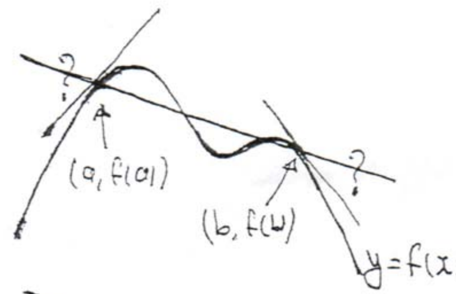
$$t(x) = \frac{\sqrt{1+x^2}}{40} + \frac{5-x}{50} .$$

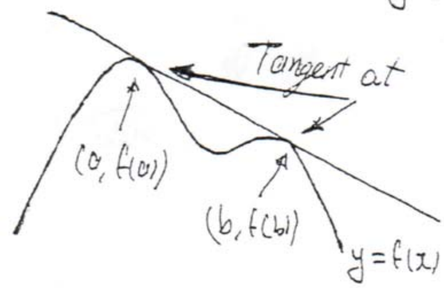
Now let's find the absolute minimum point of $t(x)$ for $0 \le x \le 5$.
`t[x_]:=Sqrt[1+x^2]/40+(5-x)/50`
`t′[x]` This gives the derivative of $t(x)$ but is not defined as a new function. You may name this derivative another function such as `tp[x]` by `tp[x_]:=t′[x]` .
Critical points are the x values in the domain of t at which $t'(x)$ is zero or undefined. Notice that $t'(x)$ is defined for all x .
`Solve[t′[x]==0, x]` In some instances, it might be necessary to use `NSolve` or `FindRoot` commands.

Since the domain $[0,\, 5]$ is a closed interval, the absolute minimum occurs either at an endpoint or at a critical point in that interval. But, in some instances, it might be necessary to consider the sign of the first derivative $t'(x)$.
`t[0]//N`
`t[4/3]//N`
`t[5]//N`
The minimum time is 0.115 hours and the motorcycle should be parked $\frac{4}{3}$ miles down the road.
`Clear[t]`

6. You can easily find a line tangent to the graph of a differentiable function at a point on its graph. However, given a differentiable function can you find a line tangent to its graph at two points, if such a line exists? Here we examine one such problem.

Consider function $y = f(x)$ with two points on its graph having $x = a$ and $x = b$ and the corresponding secant line through points $(a,\, f(a))$ and $(b,\, f(b))$. Since there is only one straight line through two points, there can be a tangent line to the graph at the points $(a,\, f(a))$ and $(b,\, f(b))$ if and only if the secant line is tangent to the graph at both $x = a$ and $x = b$.





This means that there is such a tangent line if and only if the slope of the secant line through points $(a,\, f(a))$ and $(b,\, f(b))$ is the same as the slopes of tangent lines to the graph at $x = a$ and $x = b$.

$$\frac{f(b)-f(a)}{b-a} = f'(a) \quad \text{and} \quad \frac{f(b)-f(a)}{b-a} = f'(b)$$

(a) Find a line that is simultaneously tangent to the graph of $f(x) = x^4 + 2x^3 - 2x^2$ at two points on its graph.

```
f[x_]:=x^4+2x^3-2x^2
```

A line is tangent to the graph at $x = a$ and $x = b$ if and only if the above two equations are satisfied simultaneously. Therefore, we must solve a system of equations.

```
Solve[{(f[b]-f[a])/(b-a)==f′[a], (f[b]-f[a])/(b-a)==f′[b]}, {a, b}]
```

The only pair of acceptable solution $(a \neq b)$ is $a = \frac{1}{2}(-1-\sqrt{7})$ and $b = \frac{1}{2}(-1+\sqrt{7})$.

Now find the equation of the line through points $(a,\, f(a))$ and $(b,\, f(b))$.

```
a=1/2 (-1-Sqrt[7])
b=1/2 (-1+Sqrt[7])
tl[x_]:=f[a]+((f[b]-f[a])/(b-a))(x-a)
```

We can see the simplified form of the equation of this line by evaluation the following.

```
Simplify[tl[x]]
```

To visually verify the result graph the function and this line.

```
Plot[{f[x], tl[x]}, {x, -3, 2}, PlotRange->{{-3, 2}, {-10, 5}}]
Clear[f, a, b, tl]
```

Perform each of the following exercises. It might be desirable to do some work without using Mathematica's computing capabilities (by hand). In any case, be sure to explain your work by typing in necessary comments in text cells.

A. Consider the function $f(x) = (x^4 - 3)e^{(-x^2)}$.

(i) Find the intervals on which it is increasing.

(ii) Find the intervals on which it is decreasing.

(iii) Find all its local extreme points.

(iv) Graph this function. Do the results of the last three parts match the graph?

B. Consider the function $g(x) = \frac{x^3}{3x^2+1}$.

(i) Find the intervals on which its graph is concave up.

(ii) Find the intervals on which its graph is concave down.

(iii) Find the inflection points of its graph.

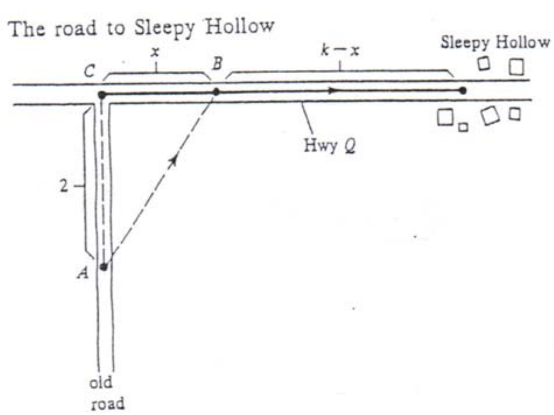
(iv) Graph this function. Are the changes of concavity seen in the graph?

C. Consider the curve defined by $x^2 \cos^2 y - \sin y = 0$.

(i) Find y' in terms of x and y .

(ii) Find y'' in terms of x , y and y' .

D. Thelma is driving north on an old road, trying to find a telephone. The road becomes impassable at A , where a sign says that it is 2 miles to Highway Q (which Thelma knows goes east for 3 or 4 miles to Sleepy Hollow). She figures that she can hike to B at $4\,mph$ and jog to Sleepy Hollow at $5\,mph$. Find x to minimize her time. (Note: k is the unknown distance from C to Sleepy Hollow. Hint: Apply the first derivative test for absolute extrema by considering the sign of the first derivative of the time function.)



E. Find the equation of a line tangent to the graph of $h(x) = x^4 + 2x^3$ at two points. Graph the function and the tangent line on the same coordinate system.