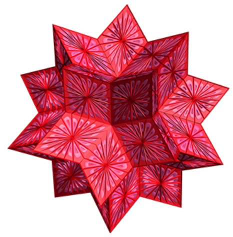
Mathematics Computer Laboratory - Math 1200 - Version 14
Lab 6 - Trigonometric Functions©



Due

You should only turn in exercises in this lab with its title and your name in `Title` and `Subtitle` font, respectively. Edit your document and use appropriate margins to get a polished and neat document.

1. This lab introduces built-in trigonometric functions and their inverses. It discusses the capabilities of Mathematica in both solving trigonometric equations and manipulating trigonometric expressions. The command `FindRoot` is used for solving trigonometric equations. Also, graphing is suggested as an alternative to algebra in establishing identities and as a way of finding starting values for the command `FindRoot`.

2. The trigonometric functions are built into Mathematica. Their default angle measure is in radians. Try the following.

   (a) `Sin[Pi/6]` This is $\sin(\pi/6) = \sin(30°)$.

   (b) To evaluate $\sin(30°)$, first convert 30 degrees to radians. $30° = 30 \times \frac{\pi}{180} = 30(0.0174533) = 0.523599$ radians. In Mathematica the constant $\frac{\pi}{180}$ is designated by `Degree`. Try each of the following in a different cell.

   `N[Degree]`
   `N[Degree]==N[Pi/180]`
   `Sin[30 Degree]` This is $\sin(30°)$.

   You may also use a multiplication sign "*" between the angle degree measure and `Degree` .

   `Cos[45*Degree]` Notice that Mathematica gives the exact value of $\cos(45°)$.
   `N[%]`

   (c) `Tan[Pi/4]` This is $\tan(\pi/4)$.

   (d) `Cot[135 Degree]` This is $\cot(135°)$.

   (e) `Sec[-Pi/4]` This is $\sec(-\pi/4)$.

   (f) `Csc[270 Degree]` This is $\csc(270°)$.

3. In graphing trigonometric functions the real numbers in the domain are interpreted as radians by Mathematica. Try the following.

   (a) `Plot[2 Sin[x+Pi/4],{x,-4Pi, 4Pi}]`

3. Continued.

(b) `Plot[3Cos[x]-2,{x,-4, 8}]` This is the graph of $y = 3\cos x - 2$ from $x = -4$ radians to $x = 8$ radians.
Notice that at $x = \pm\pi \approx \pm 3.14$, the value of the function is $y = 3\cos(\pm\pi) - 2 = -3 - 2 = -5$. Also at $x = 2\pi \approx 6.28$, the value of the function is $y = 3\cos(2\pi) - 2 = 3 - 2 = 1$. Confirm the coordinates of three points $(\pm 3.14, -5)$ and $(6.28, 1)$ by reading them off the graph. (Right click on the graph and choose Drawing Tools (or Get Coordinates) in the window that opens up. In the new window, click on the symbol that looks like a cross hairs $\boxed{-\!|\!-}$ and move the cursor over the graph.)

(c) `Plot[Tan[x], {x, -2Pi, 2Pi}, PlotRange->{{-2Pi, 2Pi}, {-10, 10}}]`

(d) `Plot[{Sin[x], 1-x^2/2}, {x, -10, 10}]` This the graph of $y = \sin x$ and $y = 1 - \frac{x^2}{2}$ on the same coordinate system.

(e) Read off and record the approximate coordinates of the two points of intersections of $y = \sin x$ and $y = 1 - \frac{x^2}{2}$ from the graph.

4. The commands `Solve` and `NSolve` can solve some trigonometric equations. However, the solutions may be restricted to certain intervals. These are the intervals the trigonometric functions are restricted to in order to define their inverses (Principal Value Branches). In other words, the solutions may be listed only in the range of appropriate inverse trigonometric function. Try the following.

(a) `NSolve[Sin[x]==0.5, x]` Mathematica finds x-value between $-\frac{\pi}{2} \approx -1.5708$ and $\frac{\pi}{2} \approx 1.5708$ such that $\sin x = 0.5$. The error message is due to the fact that it cannot find all solutions of this equation.

Can you find another solution between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? Hint: On a scratch paper hand draw a circle of radius one and remember that $\sin x$ is the y-coordinate of the point on the circle with angle x. Check your answer by evaluating `Sin[answer]`. List all solutions of this equation.

(b) `Solve[2(Cos[x])^2 -3Cos[x]+1==0, x]` Since the answers are standard angles, Mathematica can list all solutions: $x = 2n\pi$, $-\pi/3 + 2n\pi$, $\pi/3 + 2n\pi$ where n is an integer $n = 0, \pm 1, \pm 2, \cdots$. Mathematica uses `C[1]` in place of n. It is up to you to find the solution in a particular interval. List all solutions between $-\pi$ and π.

Note: In Mathematica, $\cos^2 x$ must be inputted as `(Cos[x])^2` or `Cos[x]^2`. This is the syntax for any power of any trigonometric function.

(c) `NSolve[Sin[x]==1-x^2/2, x]` As you saw earlier this equation has two solutions; however, the command `NSolve` cannot find any of the solutions. Shortly, we will discuss a method for accurately approximating solutions of this equation.

5. The output of `NSolve[Sin[x]==0.5, x]` is the value of `ArcSin[0.5]`$= \sin^{-1}(0.5)$. Therefore, Mathematica's answer to this equation is in the range of $y = \sin^{-1}(x)$ which is the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Here are the domain and range for the six inverse trigonometric functions.

5. Continued.

Function Notations			
Mathematica	Standard	Domain	Range
`ArcSin[x]`	$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
`ArcCos[x]`	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
`ArcTan[x]`	$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
`ArcCot[x]`	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$
`ArcSec[x]`	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
`ArcCsc[x]`	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Warning: The above range values are for the inverse trigonometric functions in Mathematica. Some books use different ranges for the inverse secant and cosecant functions.

(a) `ArcSin[0.5]` Is this the same value as the output for `NSolve[Sin[x]==0.5, x]`?

(b) `ArcCos[-Sqrt[3]/2]` This gives the angle between 0 and π whose cosine is $-\frac{\sqrt{3}}{2}$.

Can you find all solutions of the equation $\cos\theta = -\frac{\sqrt{3}}{2}$? Hints: Draw a circle of radius one and remember that $\cos\theta$ is the x-coordinate of the point on the circle with angle θ. Find the answer which is between π and $\frac{3\pi}{2}$, and then use the fact that the period of the function $y = \cos\theta$ is 2π.

(c) `N[ArcTan[1999]]` The output value is in radians and is close to $\pi/2$.

(d) `ArcSec[2]` Is `ArcSec[2]`$\in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$?

(e) The trigonometric expression $\sin(\cos^{-1} x)$ is one which you might have seen before and will see again in Calculus. Evaluate the following.

`Sin[ArcCos[1/2]]` You should get $\frac{\sqrt{3}}{2}$ which is the sine of the angle between 0 and π whose cosine is $\frac{1}{2}$.

`Sin[ArcCos[x]]` Therefore $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$ for $-1 \le x \le 1$.
`Cos[ArcSin[x]]` The algebraic form of $\cos(\sin^{-1} x)$ is also $\sqrt{1 - x^2}$ if $-1 \le x \le 1$.

6. When the `NSolve` command fails, the command `FindRoot` can often give an accurate approximation of a solution. The `FindRoot` command uses a numerical method for approximating solutions which requires a starting point "close" to the actual solution. An appropriate graph can give a rough estimate of a solution for use as a starting point. Try the following.

(a) `NSolve[Cos[x]==x, x]` The `NSolve` command fails to solve the equation $\cos x = x$.

(b) Preparatory to using `FindRoot`, graph $y = \cos x$ and $y = x$ to estimate solutions of the above equation, if any.

```
Plot[{Cos[x], x}, {x, -10, 10}]
```

6. Continued.

(c) Read off and record the x-coordinate of the point of intersection of the above graphs. You should get a value close to 0.8. However, in this case there is no bad estimate. (Click on the graph, hold down the $\boxed{\text{Ctrl}}$ key and then push the $\boxed{\text{d}}$ key. In the new window, click on the symbol that looks like a cross hairs $\boxed{-|-}$ and move the cursor over the graph.)

(d) `FindRoot[Cos[x]==x, {x, 0.8}]` The value 0.8 is the starting point of the numerical search for the solution in the `FindRoot` command. You should get `{x → 0.739085}`. Change the value of 0.8 slightly and evaluate it again. Do you get the same answer?

(e) Now let's try to approximate solutions of the equation $\sin x = 1 - \frac{x^2}{2}$ which we discussed earlier.

Use the approximate x-coordinate of their two points of intersections that you recorded earlier or go back to that graph and read them off now. Any values close to $x = 0.8$ and $x = -1.9$ should be good estimates.

`FindRoot[Sin[x]==1-x^2/2, {x, 0.8}]` You should get `{x → 0.774981}`.

Vary the starting value 0.8 and evaluate it again. Notice that you might not get the same answer if you use a value smaller than -0.74.

`FindRoot[Sin[x]==1-x^2/2, {x, -1.9}]`

Therefore the two solutions of the equation $\sin x = 1 - \frac{x^2}{2}$ are $x \approx 0.774981$ and $x \approx -1.96188$.

7. Although Mathematica has three commands for manipulating trigonometric expressions, its capabilities are limited and it does not always give the desired form. Try the following.

(a) `TrigExpand[Sin[2x+Pi]]` Mathematica used the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

(b) The command `TrigExpand` expands out trigonometric expressions into a sum of terms.

`TrigExpand[Sin[x]Cos[2x]]` You should get $\texttt{Cos[x]}^2\,\texttt{Sin[x]} - \texttt{Sin[x]}^3$.

(c) `TrigFactor[Cos[x]^2 Sin[x] - Sin[x]^3]` Notice that you do not get back `Sin[x]Cos[2x]`.

The command `TrigFactor` factors trigonometric expressions into products of terms, but it does not simplify them.

(d) Now, simplify the output of the last part.

`Simplify[%]` This time you should get the starting expression `Sin[x]Cos[2x]`.

(e) `TrigReduce[Cos[x]^2 Sin[x] - Sin[x]^3]` You should get $\frac{1}{2}(-\texttt{Sin[x]} + \texttt{Sin[3x]})$.

7. Continued.

(f) The command `TrigReduce` tries to replace trigonometric expressions with ones using multiple angles, but may not necessarily come up with the form you have in mind.

```
TrigReduce[Sin[x]^4]
```

8. The commands `TrigExpand`, `TrigFactor`, `TrigReduce` and `Simplify` can sometimes be used to prove trigonometric identities; more often, the only way to use Mathematica to check a trigonometric identity is through visual inspection of the graphs of each side.

(a) Show that $\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$.

```
TrigExpand[Sin[x]Cos[2x]]
```

`TrigReduce[%]` You should get $\frac{1}{2}(-\mathtt{Sin}[\mathtt{x}] + \mathtt{Sin}[\mathtt{3x}])$. Therefore $\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$ is an identity.

(b) Visually check whether or not $\cos 2x - \sin 2x = \sqrt{2}\cos(2x + \pi/4)$ is an identity.

```
Plot[{Cos[2x]-Sin[2x], Sqrt[2] Cos[2x+Pi/4]}, {x, -5, 5}]
```

If you see only one graph, it means that $\cos 2x - \sin 2x = \sqrt{2}\cos(2x + \pi/4)$ on the interval $[-5, 5]$ (assuming there were no syntax errors).

It might be better to shift one of graphs slightly to see that there are indeed two graphs.

```
Plot[{Cos[2x]-Sin[2x], Sqrt[2] Cos[2x+Pi/4] +0.1}, {x, -5, 5}]
```

If the two graphs get closer as you change 0.1 to numbers closer to zero, then $\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$ on the interval $[-5, 5]$. Therefore this is probably an identity.

(c) Visually check whether $\sin 3x - \sin x = \cos 2x \sin x$ is an identity or not.

```
Plot[{Sin[3x]-Sin[x], Cos[2x]Sin[x]}, {x, -4, 4}]
```

The equation $\sin 3x - \sin x = \cos 2x \sin x$ is not an identity since the two graphs are different.

Perform each of the following exercises. Some will require work beyond Mathematica. Include all additional work and appropriate explanatory comments (type in text cells).

A. Find the (numerical) value of each of the following.

(i) $\sec(-72°)$

(ii) $\cot(22.1)$ If an angle measure appears without "°" it is in radians.

(iii) $\cos^{-1}(-\frac{2}{3})$

(iv) $\sec^{-1}(\frac{5}{4})$ Give your answer in degrees.

(v) $\tan^{-1}(\tan\frac{3\pi}{4})$ Explain why the answer is not $\frac{3\pi}{4}$.

(vi) $\sec(\tan^{-1} 12.5)$

B. Find all solutions in the interval $[\pi, 2\pi]$ of the equation $2\sin^2 x + 3\sin x + 1 = 0$. Check your answers.

Note: First use the `Solve` command and then look for the answers which are in the interval $[\pi, 2\pi]$.

C. Use the `FindRoot` command (in combination with graphing to find starting values) to approximate all seven solutions of the equation $\frac{x}{5} = 2\sin(x + \frac{\pi}{4})$.

D. Determine whether or not each of the following equations is an identity. First perform a visual inspection. If an equation seems to be an identity, then try to use necessary Mathematica command(s) (`TrigExpand`, `TrigFactor`, `TrigReduce`, and `Simplify`) to prove it.

(i) $\sin(6x) - \sin(2x) = 2\cos(4x)\sin(2x)$

(ii) $1 - 2\sin^2(2x) = \cos(3x)\cos x + \sin(3x)\sin x$

(iii) $\cos x - \cos(3x) = 4\sin^2(x)\cos x$

(iv) $[4\cos^2 x - 1]\sin(2x) = 2\sin(3x)\cos x$

(v) $\sin(4x) + \sin(2x) = 4\sin x\cos^3 x - \sin(2x)\cos(2x)$