Mathematics Computer Laboratory - Math 1200 - Version 14
Lab 10 - Integration©

Due

You should only turn in exercises in this lab with its title and your name in `Title` and `Subtitle` font, respectively. Edit your document and use appropriate margins to get a polished and neat document.

1. This lab discusses definite integrals using the sum of areas of rectangles, or Riemann sums, which approximate areas of certain regions. Then, the definite integrals are evaluated using: the limit value of Riemann sums; the `Integrate` command; and the antiderivatives of integrands.

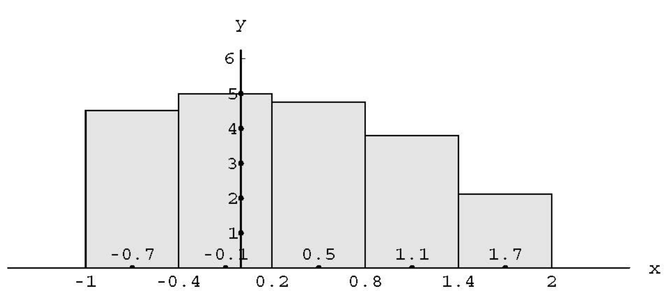
2. The `GeneralizedBarChart` command becomes available after the `BarCharts` package is loaded. Note: In Mathematica versions 7-11, you will get a message stating that the command is obsolete. But, there is no exact replacement for it and it will still work!

(a) `Needs["BarCharts`"]` The symbol ` is on the same key with ~ and in most keyboards that key is on the far left side next to the key for number 1. The usage of `GeneralizedBarChart` is as follows.

`GeneralizedBarChart[{{`center1, height1, width1`},{`center2, height2, width2`}, ⋯}]`

Each bar is designated by an ordered triple consisting of the x-coordinate of the center of its base, its height, and its width: {center, height, width}. Note: Be sure to load the package `BarCharts` before using the command `GeneralizedBarChar`.

(b) Suppose one wanted to place 5 bars over the interval $[-1, 2]$ with the following properties:
- Each has the same width.
- Their bases fill up the interval $[-1, 2]$ (the sides of the bars touch but they do not overlap and do not spill over).
- The height of each bar is 5 minus the square of the x-coordinate of its center.



(b) Continued.

The width of each bar, the x-coordinates of the sides of the bars and the height of the bars are as follows.

$$\text{width} = w = \tfrac{2-(-1)}{5} = 0.6$$

$$\begin{array}{lll}
x_0 = -1 & & \\
x_1 = -1 + 0.6 = -0.4 & c_1 = -1 + \tfrac{0.6}{2} = -0.7 & h_1 = 5 - (-0.7)^2 = 4.51 \\
x_2 = -0.4 + 0.6 = 0.2 & c_2 = -0.7 + 0.6 = -0.1 & h_2 = 5 - (-0.1)^2 = 4.99 \\
x_3 = 0.2 + 0.6 = 0.8 & c_3 = -0.1 + 0.6 = 0.5 & h_3 = 5 - 0.5^2 = 4.75 \\
x_4 = 0.8 + 0.6 = 1.4 & c_4 = 0.5 + 0.6 = 1.1 & h_4 = 5 - 1.1^2 = 3.79 \\
x_5 = 1.4 + 0.6 = 2 & c_5 = 1.1 + 0.6 = 1.7 & h_5 = 5 - 1.7^2 = 2.11
\end{array}$$

Input and evaluate the following.

```
GeneralizedBarChart[{{-0.7, 4.51, 0.6}, {-0.1, 4.99, 0.6}, {0.5, 4.75, 0.6},
                     {1.1, 3.79, 0.6}, {1.7, 2.11, 0.6}}]
```

(c) The previous bar list can be generated by formulas using the `Table` command. Notice that

$$\begin{array}{ll}
\text{width} = w = \tfrac{2-(-1)}{5} = 0.6 & \text{for } k = 1\,,\ 2\,,\ 3\,,\ 4\,,\ 5 \\
c_k = -0.7 + (k-1)0.6 & \text{for } k = 1\,,\ 2\,,\ 3\,,\ 4\,,\ 5 \\
h_k = 5 - {c_k}^2 = 5 - [-0.7 + (k-1)0.6]^2 & \text{for } k = 1\,,\ 2\,,\ 3\,,\ 4\,,\ 5\,.
\end{array}$$

Try the following.

```
GeneralizedBarChart[Table[{-0.7+(k-1)0.6, 5-(-0.7+(k-1)0.6)^2, 0.6}, {k, 1, 5, 1}]]
```

You should get that same graph as in part (b).

(d) It is easy to generalize to any number of bars with the same properties to fill the interval $[-1,\,2]$. Suppose we want to fill this interval with 10 bars. The width of each bar will be $w = \frac{2-(-1)}{10} = 0.3$ and the position of the first center is $c_1 = -1 + \frac{w}{2} = -0.85$.

$$\begin{array}{ll}
\text{width} = w = \tfrac{2-(-1)}{10} = 0.3 & \text{for } k = 1\,,\ \cdots\,,\ 10 \\
c_k = -0.85 + (k-1)0.3 & \text{for } k = 1\,,\ \cdots\,,\ 10 \\
h_k = 5 - {c_k}^2 = 5 - [-0.85 + (k-1)0.3]^2 & \text{for } k = 1\,,\ \cdots\,,\ 10
\end{array}$$

Try the following.

```
GeneralizedBarChart[Table[{-0.85+(k-1)0.3, 5-(-0.85+(k-1)0.3)^2, 0.3}, {k, 1, 10, 1}]]
```

You should get ten bars, with appropriate heights, filling the interval $[-1,\,2]$.

(e) The width of n equal-width bars covering the interval $[a,\,b]$ is $w_n = \frac{b-a}{n}$ and the position of the first center is $c_1 = a + \dfrac{w_n}{2}$. In general, for n such bars

$$\begin{array}{ll}
w_n = \tfrac{b-a}{n} & \text{for } k = 1\,,\ \cdots\,,\ n \\
c_k = (a + \tfrac{1}{2}\,w_n) + (k-1)w_n & \text{for } k = 1\,,\ \cdots\,,\ n \\
h_k = \text{a function of } c_k & \text{for } k = 1\,,\ \cdots\,,\ n\,.
\end{array}$$

2. Continued.

(f) The following uses this formulation to show 5, 10 and 15 bars with the properties stated in (b) covering the interval $[-1, 2]$. Input all of the following in one cell as written and evaluate it.

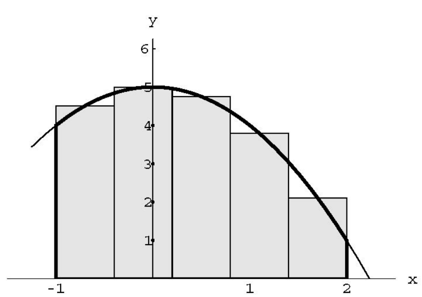
```
w[n_]:=(2-(-1))/n;
c[k_]:=(-1+(1/2)*w[n])+(k-1)w[n];
h[k_]:=5-c[k]^2;
Table[GeneralizedBarChart[Table[{c[k], h[k], w[n]},{k, 1, n}]],{n, 5, 15, 5}]
```

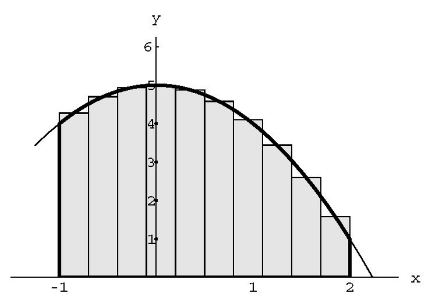
3. Our interest in such bar charts is to illustrate definite integrals as the limiting value of Riemann sums since, for nonnegative integrand functions, definite integrals represent the area under a graph. From the following graphs, we see that increasing the number of bars results in the bars more accurately representing the area under the function $f(x) = 5 - x^2$ over the interval $[-1, 2]$.

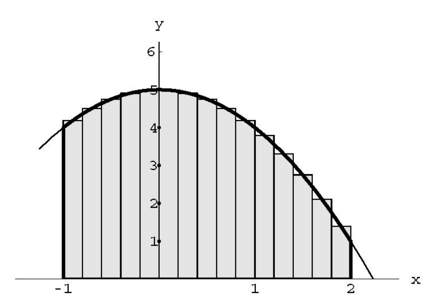
(a) You can visually compare the area under the function and the sum of areas of the bar by doing the following.

```
n=5; bars=GeneralizedBarChart[Table[{c[k], h[k], w[n]},{k, 1, n}]]
fgraph=Plot[5-x^2, {x,-1.5,2.5}]
Show[bars, fgraph]
Clear[bars, fgraph, n]
```

Repeating these with $n = 10$ and $n = 15$ shows how the approximation improves as n gets larger.







For each bar, the area is:

$$\begin{array}{ll} \text{Area} = \text{width} \times \text{height} & \text{for } k = 1, \cdots, n \\ A_k = w \times h_k = \frac{2-(-1)}{n} \times (5 - {c_k}^2) & \text{for } k = 1, \cdots, n. \end{array}$$

These areas can be added using the `Sum` command.

(b) Input the following in one cell and evaluate it to find the sum of the areas of the 5 bars ($n = 5$) in the left graph.
`n=5; Sum[w[n]*h[k], {k, 1, n}]//N` You should get 12.09. The command `//N` causes the sum to be given in decimal form.

The usage of the command `Sum` is as follows.

`Sum[`expression, {index, start value, end value, step size}]

(c) Input the following in one cell and evaluate it to find the sum of the areas of the 10 bars ($n = 10$) in the middle graph.
`n=10; Sum[w[n]*h[k], {k, 1, n}]//N` You should get 12.0225.

3. Continued.

(d) Input the following in one cell and evaluate it to find the sum of the areas of the 15 bars ($n = 15$) in the right graph.
`n=15; Sum[w[n]*h[k], {k, 1, n}]//N` You should get 12.01 .

(e) Notice that as the number of the bars increases (or their width decreases), this sum gets closer to 12 .
`Table[{n, Sum[N[w[n]*h[k]], {k, 1, n}]}, {n, 5, 50, 5}]//TableForm`

Sometimes one can find the limit value of Riemann sums using the `Limit` command.
`Clear[n]; Limit[Sum[N[w[n]*h[k]], {k, 1, n}], n->Infinity]` The `Limit` command will not work for every function.

(f) The definite integral of a function $f(x)$ over an interval $[a, b]$ is defined as the limit of the Riemann sums defined earlier, if such a limit exists:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} w f(c_k)\,.$$

If an integrable function is nonnegative over an interval, then the area under the graph of the function over that interval is the definite integral of the function over that interval. For example, for $f(x) = 5 - x^2$:

$$\int_{-1}^{2} (5 - x^2)\,dx = \text{Area under } f(x) \text{ over the interval } [-1,\, 2]\ .$$

4. Evaluate the following indefinite integrals using the `Integrate` command or appropriate integration symbols from a palette.

(a) `Integrate[2x^3+x-1, {x, 1, 3}]` This is $\int_1^3 (2x^3 + x - 1)\,dx$. Check the answer by evaluating this integral by hand.

(b) `Integrate[Cos[x]^2, {x, -Pi/2, Pi/3}]` This is $\int_{-\pi/2}^{\pi/3} \cos^2 x\,dx$.

(c) `NIntegrate[Cos[x]^2, {x, -Pi/2, Pi/3}]` This gives the approximate decimal value of the last integral. Are the two values in parts (b) and (c) equal?

(d) `Clear[w, c, h]`

5. Evaluate $\int_0^{2\pi} x\sin x\,dx$ using both the limit of the Riemann sums and the `Integrate` command.

(a) We can find the value of the Riemann sums as n gets large as follows.

```
w[n_]:=(2 Pi-0)/n;
c[k_]:=(0+(1/2)*w[n])+(k-1)w[n];
h[k_]:=c[k]*Sin[c[k]];
Table[{n, Sum[N[w[n]*h[k]], {k, 1, n}]}, {n, 10, 100, 10}]//TableForm
```

Notice the values of the sum get close to -6.284 .
`Limit[ Sum[N[w[n]*h[k]], {k, 1, n}], n->Infinity]` Notice that Mathematica does not give a numerical answer. However, Mathematica is capable of finding this limit if additional instructions are given. But we will not discuss that at this time.

5. Continued.

(b) We can evaluate this definite integral as follows.
`Integrate[x Sin[x], {x, 0, 2 Pi}]` Are the answers in parts (a) and (b) equal?

(c) `Clear[w, c, h]`

6. Recall that an indefinite integral represents any antiderivative of the integrand.

$$\int f(x)\,dx = F(x) + C \text{ where } F'(x) = f(x)$$

We can also use Mathematica to find indefinite integrals.

(a) `Integrate[2x^3+x-1, x]` This is $\int (2x^3+x-1)\,dx$. The `x` after the comma, represents the integration variable. Notice that Mathematica does not give the constant C and so states only one antiderivative.

(b) To verify the answer, we must show that $F'(x) = f(x)$.
`D[%, x]` You should get back $2x^3 + x - 1$.

(c) `Integrate[Cos[x]^2, x]` This is $\int \cos^2 x\,dx$.

(d) `Integrate[(2x-1)Sqrt[x^2-x+5], x]` You should check this by hand using the u-substitution.

(e) Mathematica is not capable of applying the u-substitution in some seemingly simple cases, as is seen in the next example.
`Integrate[2x(x^2+1)^100, x]` Not a very palatable answer. Is it?
Find this integral, $\int 2x(x^2+1)^{100}\,dx$, by hand using the u-substitution.

7. A definite integral can also be evaluated using the antiderivative (or the indefinite integral) of the integrand:

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F'(x) = f(x)\,,\ \text{or } F(x) = \int f(x)\,dx\,.$$

(a) `F[x_]=Integrate[2x^3+x-1, x]` Notice that we have only used the equality sign " = " and not " := ". This results in `F[x]` being the results of the integration and not the integral itself.

(b) `F[x]` You should obtain the antiderivative $\frac{1}{2}x^4 + \frac{1}{2}x^2 - x$ and not `Integrate[2x^3+x-1, x]`.

(c) `F[3]-F[1]` You should get the same value as $\int_1^3 (2x^3 + x - 1)\,dx$ obtained in 4(a).

(d) `Clear[F]`

Perform each of the following exercises. It might be desirable to do some work without using Mathematica's computing capabilities (by hand). In any case, be sure to explain your work by typing in necessary comments in text cells.

A. Suppose the bars in each part have equal width and cover the interval $[1, 5]$, with no overlap or spill over. Let the height of each bar be the reciprocal of the x-coordinate of its center ($h_k = \frac{1}{c_k}$). For each given number of bars, graph them along with the function $f(x) = \frac{1}{x}$ on the same coordinate system.

(i) Five bars.

(ii) 10 bars.

(iii) 15 bars.

Use the `Sum` command to add up the areas of these bars. Based on these areas, estimate the value of $\displaystyle\int_1^5 \frac{1}{x}\,dx$.

B. Evaluate $\displaystyle\int_{-1}^{2} \frac{1}{x^2+x+1}\,dx$ in three ways and compare your answers.

(i) Using the `Table` and `Sum` commands to estimate the value of the appropriate Riemann sum.

(ii) Using the `Limit` command to evaluate the appropriate Riemann sum.
Note: Use or placement of `N` commnad might make a difference.

(iii) Using the `NIntegrate` command for definite integrals.

C. Evaluate the following using the `Integrate` command or by hand if hand evaluation results in a "better" answer.

(i) $\displaystyle\int_{-1}^{2} \frac{x}{\sqrt{13-3x^2}}\,dx$

(ii) $\displaystyle\int_{2}^{3} \sqrt{9-x^2}\,dx$ In addition to its exact value, find its approximate value using the NIntegrate command.

(iii) $\displaystyle\int_{2.7}^{-4.1} e^x \cos x\,dx$ Re-evaluate this integral using $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$. Are the two answers the same?

(iv) $\displaystyle\int x^2 \sin x\,dx$ Check your answer by showing that its derivative is the integrand.

(v) $\displaystyle\int (3x^2+1)(x^3+x-2)^{10}\,dx$ Does the hand evaluation give a "better" result? If yes, show it.