Two Finite Inverse Hilbert Transform Formulae for Region-of-Interest Tomography

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ABSTRACT: Recently we published two explicit formulae for finite inverse Hilbert transforms (You and Zeng, Inv Probl 22 (2006), L7-L101). This paper presents a straightforward proof of the formulae, the data requirements, and some computer simulations to verify the formulae. Examples of region-of-interest tomography using truncated projections and the first formula of the finite inverse Hilbert transform are shown.

I. INTRODUCTION
There is increased interest in region-of-interest (ROI) reconstruction with truncated projections (Noo et al., 2004; Rullgård, 2004; Zhuang et al., 2004; Zou and Pan, 2004; Pack et al., 2005; Sidky and Pan, 2005). The state-of-the-art method for ROI reconstruction with truncated data is the filtering of differentiated-backprojection (FDB) algorithm. This algorithm consists of following two steps (Noo et al., 2004): (i) backprojecting the derivative of the data and (ii) evaluating a one-dimensional (1D) finite inverse Hilbert transform row-by-row. Let \( f(t) \) be a smooth function and have a finite support of \([-q, q]\) and \( F(s) \) be its Hilbert transform, then a Hilbert transform pair is given as

\[
\begin{align*}
F(s) &= \text{pv} \int_{-q}^{q} \frac{f(t)}{\pi(s-t)} \, dt, \quad (1) \\
f(t) &= \text{pv} \int_{-\infty}^{\infty} \frac{F(s)}{\pi(t-s)} \, ds, \quad (2)
\end{align*}
\]

where \( \text{pv} \) indicates the principle value of the integral. Since the function \( f(t) \) has a finite support, the inverse Hilbert transform (2) can also be obtained with finite data using a classic finite inverse Hilbert transform formula (Polyanin and Manzhirov, 1998):

\[
f(t) = \frac{-1}{\sqrt{q^2 - t^2}} \left( \int_{-q}^{q} \frac{\sqrt{q^2 - s^2}}{\pi(t-s)} F(s) \, ds + C \right), \quad (3)
\]

where \( C \) is an arbitrary constant. Thus, the finite inverse Hilbert transform of \( F(s) \) is not unique. It was shown in Tricomi (1985) that the integral of \( f(t) \) obtained by (3) over \([-q, q]\) is

\[
\int_{-q}^{q} f(t) \, dt = -\pi C. \quad (4)
\]

Therefore, formula (3) is sensitive to the selection of the constant \( C \). According to (4), a small error in \( C \) may cause large errors in recovering the original \( f(t) \) from \( F(s) \). The next section will use this classic formula (3) to rederive the two formulae presented in You and Zeng (2006).

The finite inverse Hilbert transform has been studied thoroughly in literature (Noo et al., 2004; Rullgård, 2004; Zhuang et al., 2004; Zou and Pan, 2004; Pack et al., 2005; Sidky and Pan, 2005; You and Zeng, 2006), specially in books on integral equations (Tricomi, 1985; Polyanin and Manzhirov, 1998). The contribution of this current note is threefold: presenting a straightforward proof of the two formulae presented in You and Zeng (2006), providing numerical examples to verify these two formulae, and clearly stating the projection data requirements when these formulae are used in ROI image reconstruction. We would like to make it clear that this is not the first paper to apply finite inverse Hilbert transform to ROI image reconstruction, many previous works (Noo et al., 2004; Rullgård, 2004; Zhuang et al., 2004; Zou and Pan, 2004; Pack et al., 2005; Sidky and Pan, 2005) should deserve the credit.
II. FORMULA DERIVATION

Let \( R > r \geq q \). From (3), we have

\[
- f(t) \sqrt{R^2 - t^2} = \int_{-R}^{R} \frac{\sqrt{R^2 - s^2}}{\pi(t - s)} F(s) \, ds + C, \tag{5}
\]

\[
- f(t) \sqrt{r^2 - t^2} = \int_{-r}^{r} \frac{\sqrt{r^2 - s^2}}{\pi(t - s)} F(s) \, ds + C. \tag{6}
\]

We require that the same \( C \) be used in (5) and (6), and

\[
\int_{-R}^{R} f(t) \, dt = \int_{-r}^{r} f(t) \, dt = \int_{-q}^{q} f(t) \, dt = -\pi C.
\]

This selection of \( C \) guarantees that \( f(t) \) has a finite support of \([-q, q]\) and that the recovered function \( f(t) \) is unique.

Subtracting (5) from (6) yields

\[
f(t)(\sqrt{R^2 - t^2} - \sqrt{r^2 - t^2}) = -\int_{-R}^{R} \frac{\sqrt{R^2 - s^2}}{\pi(t - s)} F(s) \, ds + \int_{-r}^{r} \frac{\sqrt{r^2 - s^2}}{\pi(t - s)} F(s) \, ds.
\]

Let

\[
k(R, r, t) = \begin{cases} 
- \frac{\sqrt{R^2 - t^2}}{\pi(r - s)} & r < |t| < R, \\
- \frac{\sqrt{r^2 - t^2}}{\pi(r - s)} + \frac{\sqrt{R^2 - t^2}}{\pi(r - s)} & 0 \leq |t| < r, \\
0 & \text{otherwise}.
\end{cases}
\]

then (7) becomes

\[
f(t) = \frac{1}{\sqrt{R^2 - t^2} - \sqrt{r^2 - t^2}} \int_{-R}^{R} k(R, r, t) F(s) \, ds, \tag{9}
\]

which is the first of the two formulae derived in You and Zeng (2006).

Next, we consider a special example of \( f(t) = \delta(t) \), then \( F(s) = \frac{1}{\pi} \delta(s) \). For this special case and any \( R > r > 0 \), (9) becomes

\[
\delta(t) = \frac{1}{\sqrt{R^2 - t^2} - \sqrt{r^2 - t^2}} \int_{-R}^{R} k(R, r, s) \, ds = \frac{1}{R - r} \int_{-r}^{r} k(R, r, s) \, ds \text{ for } |t| < r. \tag{10}
\]

The improper integral in Eq. (10) contains two singularities: \( s = t \) and \( s = 0 \). The improper integral is understood to mean

\[
\int_{s(t = s)}^{1} \frac{1}{s(t - s)} k(R, r, s) \, ds \equiv \lim_{\epsilon \to 0} \int_{-R}^{R} \frac{s(s - t)}{(s^2 + \epsilon^2)(s(t - s)^2 + \epsilon^2)} k(R, r, s) \, ds. \tag{11}
\]

Using Eq. (10), we can readily obtain the second finite inverse Hilbert transform formula in You and Zeng (2006) as follows. For \( R > r \) and \(-r < t < r\),

\[
f(t) = f(t) \ast \frac{1}{R - r} \int_{-r}^{r} k(R, r, s) \, ds
\]

\[
\frac{1}{R - r} \int_{-r}^{r} \frac{k(R, r, s)}{s} \int_{-r}^{r} f(y) \, dy \, ds.
\]

\[
= \frac{1}{R - r} \int_{-r}^{r} \frac{k(R, r, s)}{s} \int_{-r}^{r} \frac{f(y)}{\pi(s - y)} \, dy \, ds
\]

\[
= \int_{-r}^{r} k(R, r, s) F(t - s) \, ds. \tag{12}
\]

The changing of the order of integrals in (12) is justified by the definition (11) for a double-singularity improper integral. It is shown in the Appendix that using (11) is equivalent to using the formula of Poincare-Bertrand (Muskhelishvili, 1953), which deals with the change of the order of two Cauchy integrals.

III. DATA REQUIREMENTS

In formula (9), \( F(s) \) is required over a range \([-R, R]\) which is a little larger than that of the support of \( f(t), [-q, q] \). For example, if the support of \( f(t) \) is \([-1, 1]\), then \( r \) can be chosen as 1 and \( R \) can be chosen as 1.1. We can use \( F(s) \) over \([-1, 1.1]\) to recover \( f(t) \).

In derivation of formula (12), we have used the convolution relation of \( f(t) = f(t) \ast \delta(t) \). If the support of \( f(t) \) is \([-q, q]\), then the locally defined delta-function with (10) needs to be valid for \( |t| < 2q \). Thus \( r \) can be chosen as \( 2q \) in (10) and \( R \) should be a little larger than \( r \). Using (12), \( F(s) \) needs to be known on \([-q + R, (q + R)]\). For example, if the support of \( f(t) \) is \([-1, 1]\), then \( r \) can be chosen as 2 and \( R \) can be chosen as 2.1. Hence, we can use \( F(s) \) over \([-3.1, 3.1]\) to recover \( f(t) \). Therefore, using (9) requires fewer data for \( F(s) \) than using (12) in evaluating \( f(t) \). It is expected that (9) may be more useful than (12) in practice.

IV. NUMERICAL EXAMPLES

This section uses the following Hilbert transform pair to test the two formulae developed earlier:

\[
f(t) = \sqrt{1 - t^2} \text{ for } |t| < 1; f(t) = 0 \text{ otherwise.} \tag{13}
\]

\[
F(s) = s - \sqrt{s^2 - 1} \text{sgn}(s) \text{ for } |s| > 1; F(s) = s \text{ otherwise.} \tag{14}
\]

The numerical results are shown in Figure 1. The numerical implementation code was written in Matlab. The Figure 1a shows \( F(s) \) on \([-3.1, 3.1]\). Figure 1b shows two overlapped curves: one is \( f(t) \) given by (13) and the other is the result from (9), where \( q = 1 \) and \( R = 1.1 \). Data \( F(s) \) from \([-1.1, 1.1]\) were used to verify (9). Figure 1c shows two overlapped curves: one is \( f(t) \) given by (13) and the other is the result from (12), where \( q = 1, r = 2, \) and \( R = 2.1 \). Data \( F(s) \) from \([-3.1, 3.1]\) were used to verify (12). In all numerical integrations, the trapezoid-rule was used, and the sampling interval was 0.01.

V. EXAMPLES OF IMAGE RECONSTRUCTION

A typical FDB algorithm for image reconstruction consists of two steps (Noo et al., 2004): (i) backprojecting the derivative of the projection data (i.e., the Radon transform of the object) and (ii) evaluating a 1D inverse Hilbert transform row-by-row. The projections

\[
\begin{align*}
\text{(a) The Hilbert transform } F(s) \text{ defined in (14), (b) Overlapped curves of the original } f(t) \text{ and the result from the first formula, (c) Overlapped curves of the original } f(t) \text{ and the result from the second formula.}
\end{align*}
\]
were generated analytically at 400 views over a range of 180°. There were 513 detector bins on the detector. The "row" orientation is in the 90° direction. The derivative operation in step (i) of the algorithm was implemented as a two-point central difference. In step (ii) we chose (9) instead of (12) to perform the finite inverse Hilbert transform, because (9) requires fewer data points than (12). (This may also save a lot of computations in the backprojection.) The reconstructed image was stored in a 513 × 513 array.

For untruncated noiseless and Poisson noise corrupted projection data, computer simulations using the FDB algorithm and (9) with different ratios of \( R/r \) are shown in Figure 2. The Shepp-Logan phantom (which is also a built-in function in Matlab) was accurately reconstructed. In these simulations with nontruncated data, the parameter \( R \) was chosen as 256, i.e., half of the image dimension, and two different values of \( r \), 240 and 248, were used. The

Figure 2. FDB reconstructions with complete data using (9). (a) Noiseless data, \( R/r = 1.0667 \). (b) Noiseless data, \( R/r = 1.0323 \). (c) Noisy data, \( R/r = 1.0667 \). (d) Noisy data, \( R/r = 1.0323 \). The profiles are drawn along the horizontal central row of each image, compared with the true. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

Figure 3. FDB region-of-interest reconstructions with truncated data using (9). (a) Noiseless data, \( R/r = 1.0667 \). (b) Noiseless data, \( R/r = 1.0323 \). (c) Noiseless data, \( R/r = 1.0079 \). (d) Noisy data, \( R/r = 1.0667 \). (e) Noisy data, \( R/r = 1.0323 \). (f) Noisy data, \( R/r = 1.0079 \). The two parallel broken lines shown in (a) are defined in Figure 4. The profiles are drawn along the horizontal central row of each image, compared with the true. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]
reconstruction was not sensitive to the choices of \( r \) as long as \( R > r > q \). The parameter \( q \) represents the phantom boundary. In this Shepp-Logan phantom, the value of \( q \) varies between 138 and 184.

Figure 3 shows ROI reconstructions using truncated noiseless and noisy projection data. Only the center 313 bins out of the total 513 bins on the detector were used to acquire the projection data. In this case, the selection of parameter \( R \) is illustrated in Figure 4. Once \( R \) was determined, \( r \) was chosen according to three fixed ratios \( R/r \). The parameters \( R \) and \( r \) were different for each row of the image. The results shown in Figure 3 indicate that \( r \) can be chosen to be very close to \( R \), and the ROI reconstruction is stable.

VI. A PHANTOM EXPERIMENT

A Jaszzczak phantom was used in an experiment. The phantom was filled with 34 mCi of Tc-99m. Three low-energy-high-resolution (LEHR) parallel-hole collimators were mounted on a Philips IRIX SPECT system. The projections were acquired in a 256 \( \times \) 256 \( \times \) 120 array, with 120 views over 360°. Figure 5 shows reconstructions with the FDB algorithm and the conventional FBP algorithm, respectively. It is observed that the reconstruction with the FDB algorithm had better contrast than the reconstruction with the FBP algorithm. This may be caused by the different frequency bandwidths in the filtering procedures in these two algorithms.

VII. DISCUSSION

There are two advantages of our two formulae (9) and (12) over the classic formula (3): First, the constant \( C \) which is not easy to estimate especially when the data are noisy, should be exactly known when using (3); however, this constant is not required when using (9) and (12). Formulae do exist to determine the constant \( C \) (Noo et al., 2004). Avoiding evaluating \( C \) may not automatically give a more stable algorithm. In fact, many formulae (Polyanin and Manzhirov, 1998) exist that do not require the constant \( C \), for example,

\[
f(t) = \frac{1}{\pi} \sqrt{(1 + t) / (1 - t)} \int_{-1}^{1} \sqrt{(1 - s) / (1 + s)} F(s) / (t - s) \, ds.
\]

However, those formulae contain a singularity at least at one end of the interval on which \( f(t) \) is supported.

Second, the new formulae are stable; on the other hand, there are singularities at \( t = \pm q \) in (3). Of course, one can replace \( q \) by \( R \) (as long as \( R > q \)) in (3), so that the singularities are outside the image support to make (3) stable for image reconstruction.

Computer simulations show that \( R \) can be chosen at the available data’s boundary, and \( r \) can be chosen very close to \( R \). The second of the new formulae is in the form of a convolution, but it requires data for \( F(s) \) from \([-R+q, (R+q)]\) with \( R > r \geq q \) to recover \( f(t) \). The first of the new formulae is not in the form of a convolution, and it requires data for \( F(s) \) only from \([-R, R]\) with \( R > r \geq q \).

APPENDIX

If the formula of Poincare-Bertrand is to be used, the improper integral in the left-hand side of (11) is defined as

\[
\int_{-R}^{R} \frac{1}{s(t-s)} k(R, r, s) \, ds \equiv \frac{1}{t} \int_{-R}^{R} \left( \frac{1}{s} + \frac{1}{s-t} \right) k(R, r, s) \, ds \quad \text{(A1)}
\]

This definition implies that when \( t = 0 \), the integrals in (11) must be zero. With this definition, Eq. (10) should be written as

\[
\delta(t) = \frac{1}{R-r} \int_{s=R}^{R} k(R, r, s) \, ds + \delta(t) \quad \text{for } |t| < r \quad \text{(A2)}
\]

and (12) becomes

\[
f(t) = \frac{1}{R-r} \int_{s=R}^{R} \int_{y=-r}^{r} \int_{s=y-R}^{s+y} \frac{k(R, r, s)}{\pi s(t-s-y)} \, dy \, ds \, f(t).
\]

Using the formula of Poincare-Bertrand, the earlier equation becomes

\[
f(t) = \pi^2 \frac{k(R, r, 0) f(t)}{R-r} + \frac{1}{R-r} \int_{s=R}^{R} \frac{k(R, r, s)}{s} \int_{y=-r}^{r} \frac{f(y)}{\pi(t-s-y)} \, dy \, ds \, f(t) \quad \text{(A4)}
\]

Since \( k(R, r, 0) = (r-R)/r \),

\[
f(t) = \frac{1}{R-r} \int_{s=R}^{R} \frac{k(R, r, s)}{s} \int_{y=-r}^{r} \frac{f(y)}{\pi(t-s-y)} \, dy \, ds
\]

\[
= \int_{s=R}^{R} \frac{k(R, r, s)}{R-r} F(t-s) \, ds \quad \text{(A5)}
\]

which is the same as (12).
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