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Study of noise propagation and the effects of insufficient numbers of projection angles and detector samplings for iterative reconstruction using planar-integral data

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A rotating slat collimator can be used to acquire planar-integral data. It achieves higher geometric efficiency than a parallel-hole collimator by accepting more photons, but the planar-integral data contain less tomographic information that may result in larger noise amplification in the reconstruction. Lodge evaluated the rotating slat system and the parallel-hole system based on noise behavior for an FBP reconstruction. Here, we evaluate the noise propagation properties of the two collimation systems for iterative reconstruction. We extend Huesman’s noise propagation analysis of the line-integral system to the planar-integral case, and show that approximately 2.0 (D/dp) SPECT angles, 2.5 (D/dp) self-spinning angles at each detector position, and a 0.5 dp detector sampling interval are required in order for the planar-integral data to be efficiently utilized. Here, D is the diameter of the object and dp is the linear dimension of the voxels that subdivide the object. The noise propagation behaviors of the two systems are then compared based on a least-square reconstruction using the ratio of the SNR in the image reconstructed using a planar-integral system to that reconstructed using a line-integral system. The ratio is found to be proportional to \( \sqrt{F/D} \), where F is a geometric efficiency factor. This result has been verified by computer simulations. It confirms that for an iterative reconstruction, the noise tradeoff of the two systems is not only dependent on the increase of the geometric efficiency afforded by the planar projection method, but also dependent on the size of the object. The planar-integral system works better for small objects, while the line-integral system performs better for large ones. This result is consistent with Lodge’s results based on the FBP method. © 2006 American Association of Physicists in Medicine. [DOI: 10.1118/1.2266270]

Key words: rotating-slat collimator, noise analysis, iterative reconstruction, data sufficiency

I. INTRODUCTION

The low efficiency of the conventional gamma camera using line-integral projection data acquired with a parallel-hole collimator is one of the factors that limit the performance of the SPECT system.\(^1,2\) Since the parallel-hole collimator only allows the photons that strike orthogonally to the surface of the detector to pass through, it approximately integrates the activities over parallel lines. One way to improve its poor geometric efficiency is to use planar-integral projection data that can be acquired, for example, by a rotating CdZnTe strip gamma camera with a parallel slat collimator.\(^1,3\) The major advantage of the rotating slat collimator is its higher acceptance solid angle to the photon flux than the parallel-hole collimator. A typical parallel-hole collimator has a geometric efficiency (defined as the percentage of detected to emitted photons) in the order of 0.01%,\(^2\) while the geometric efficiency of the rotating slat collimator is in the order of 0.1%.\(^2\) Using planar-integral data, one can obtain exact reconstructions with truncated data,\(^4\) which is a valuable property for imaging large objects that cannot be adequately measured due to a small detector size. In addition, since the cost of a SPECT system is largely determined by the size of the detector, and a strip detector can be smaller than a conventional Anger camera, the strip system may be financially attractive.

However, a rotating strip detector needs an additional spinning motion compared to a regular parallel-hole detector in order to acquire a complete dataset, and the image reconstruction for a planar data detector requires a more complicated algorithm or a larger number of iterations than regular line-projection detectors. Since more deconvolutions are generally needed in the reconstruction, a higher geometric efficiency does not guarantee a better noise property in the reconstructed image.

Lodge\(^5\) evaluated and compared the rotating slat system with a conventional parallel-hole system (a line-integral system) based on the noise propagation behavior of a filtered-backprojection (FBP) reconstruction. The noise behavior tradeoff of the two systems was evaluated by the ratio of the SNR in the image reconstructed using the planar-integral system to that using the line-integral system. It was shown that the noise tradeoff of the two systems was dependent, not only on the increase in geometry efficiency resulting from the use of planar-integral data, but also on the size of the object. In SPECT, iterative methods provide more accurate reconstructions than analytical methods (e.g., an FBP method) due to their ability to incorporate various physical models. The noise propagation behavior of an iterative reconstruction for the line-integral system, and the effects of a...
finite number of data points on the noise propagation behavior of the reconstruction have been analyzed in Huesman's work. In our work, we first extend Huesman's analysis to the planar-integral case: we evaluate the noise propagation behavior of an iterative reconstruction using planar-integral data and the effects of insufficient measurements on noise propagation using the geometry of a planar-integral system; we then couple it with Huesman's analysis to determine the noise behavior tradeoff of the two systems for an iterative reconstruction, which is presented using the same image SNR ratio adopted in Lodge's work.

II. METHODS

A. Acquisition geometries

For the line-integral system, three dimensional (3D) image reconstruction can be accomplished by a slice-by-slice reconstruction approach. The 3D image region is divided into a series of parallel slices. Each row of the collimated detector defines a single slice of the 3D image. The measured projection data in this row only contribute to the reconstruction of one slice and each slice is reconstructed separately. The final 3D image is formed by stacking all the reconstructed slices. Therefore, 3D imaging with line-integral data is actually accomplished using a series of 2D reconstructions. Figure 1 shows the geometry of the line-integral system. For a 2D reconstruction (shown in Fig. 2), images are usually represented by pixels. The weight of each pixel in each projection is determined by the length made by the projection path intersecting with the pixel. As discussed in Huesman's work, the numbers of projection angles and detector sampling intervals affected the noise amplification. It was shown that a detector sampling interval (i.e., the detector bin size) smaller than 0.5_dp, and more than 1.5D_dp projection angles are required, where dp is the pixel size and D represents the diameter of the object.

For the planar-integral system, the slat collimator only collimates the strip detector in one dimension. Compared with the line-integral system, more photons can reach the detector and a higher geometry efficiency is expected. However, in order to obtain an adequate data set for the 3D reconstruction, the acquisition of the projection data requires two motions: gantry rotation about the transaxial axis (the z-axis) of the imaged object (also required for the line-integral system) and spinning of the detector about its central axis at each gantry position (not required for the line-integral system). Figure 3 shows the data acquisition geometry of the planar-integral system. Each detector position is defined by two angle parameters (θ: the gantry rotation angle and φ: the detector spinning angle). At each detector position, the slats define a number of projection planes that are parallel to the slats and normal to the detector surface. Only the photons traveling parallel to these planes can be measured by the detector. Therefore, unlike the line-integral system mentioned above, the 2D detector measures 1D projections that are approximate planar integrals over the projection planes at each detector position.

Generally, a 3D image is represented by voxels which are dp×dp×dp cubes, as shown in Fig. 4. Each projection is approximately the integral over the projection plane within
the image region. The weight of each voxel is the intersection area made by the projection plane and the voxel.

**B. Reconstruction and noise propagation analysis**

The basic idea of an iterative reconstruction is to provide a statistical estimation of the solution of a set of equations:

$$Ax = p,$$

where $A$ represents the system matrix that models the projection operator, $p$ is the measured projection data, and $x$ is the unknown emission distribution in the vector form. For the planar-integral system, the measured planar-integral value $p_i$ at the $i$th detector bin can be expressed in a discrete form in terms of the unknown image value $x_j$ of the $j$th voxel and the system noise.

$$p_i = \sum_{j=1}^{n} a_{ij} x_j + \text{noise},$$

where $a_{ij}$ is the intersection area made by the $i$th projection path and the $j$th voxel, and $n$ denotes the total number of image elements (voxels) of the object. Here the object is assumed to be a uniform sphere. The backprojection $B_k$ of the $k$th voxel is given by

$$B_k = H \sum_{i=1}^{m} p_i a_{ik} = H \sum_{i=1}^{m} a_{ik} a_{ij} x_j,$$

where $m$ is the total number of integral paths, and $m = m_x \times m_y \times m_\theta$ where $m_x$ is the number of detector bins, $m_\theta$ denotes the number of rotation angles (see Fig. 3). Here, $H$ denotes a normalization factor. The choice of $H$ is explained later. The back-projection matrix $B$ can be simplified by defining a symmetric matrix $Q$ whose elements are given by

$$Q_{kj} = H \sum_{i=1}^{m} a_{ik} a_{ij}.$$

Thus, Eq. (3) can be rewritten as

$$B_k = \sum_{j=1}^{n} Q_{kj} x_j$$

so that the backprojection vector is just the image vector multiplied by the matrix $Q$. A diagonal element of $Q$ is in the form of

$$Q_{jj} = H \sum_{i=1}^{m} a_{jj}^2,$$

but the fraction of planar-integral projections that intersect an arbitrary voxel (e.g., the $j$th voxel) with nonzero areas at each detector position $(\theta, \phi)$ is about $dp/D$ [only about $(m \times dp)/D$ of the planar integrals intersect the $j$th voxel] and when nonzero, $a_{jj}$ is approximately equal to $dp \times dp$. Thus, the normalization factor $H$ that makes the matrix $Q$ roughly independent of the imaging geometry and the number of the planar integrals is chosen by setting the $Q_{jj}$ roughly equal to 1, implying that

$$H = \frac{1}{\frac{m \times dp}{D} \left(dp \times dp\right)^2} = \frac{D}{m \times dp^5}.$$  

(7)

Substituting $H = [D/(m \times dp^5)]$ into Eq. (4) gives

$$Q_{kj} = \frac{D}{m \times dp^5} \sum_{i=1}^{m} a_{ik} a_{ij}.$$  

(8)

Thus, according to Eq. (5), a normalized least-squares solution of Eq. (1) can be expressed in terms of the inverse (or generalized inverse) of the matrix $Q$. By substitution of $B_k = H \sum_{i=1}^{m} p_i a_{ik}$ and $H = D/(m \times dp^5)$, the least-squares solution can be expressed as

$$x_j = \sum_{k=1}^{n} Q_{jk}^{-1} B_k = \sum_{k=1}^{n} Q_{jk}^{-1} \left(\frac{D}{m \times dp^5} \sum_{i=1}^{m} p_i a_{ik}\right)$$

$$= \frac{D}{m \times dp^5} \sum_{i=1}^{m} p_i \sum_{k=1}^{n} Q_{jk}^{-1} a_{ik},$$

(9)

where $Q^{-1}$ indicates the inverse of the matrix $Q$, and $D/(m \times dp^5)$ is factored out to make the following analysis more transparent. Note that in Eq. (9), only $x_j$ and $p_i$ are random variables. Since the measurements of different planar integrals are statistically independent, the variance of $x_j$ and the sum of the variance of the contributions from each projection $p_i$ added in quadrature can be related by taking variances of both the left and right sides of Eq. (9), which gives

$$\sigma^2(x_j) = \left(\frac{D}{m \times dp^5}\right)^2 \sum_{i=1}^{m} \sigma^2(p_i) \left(\sum_{k=1}^{n} Q_{jk}^{-1} a_{ik}\right)^2$$

$$= \left(\frac{D}{m \times dp^5}\right)^2 \sum_{i=1}^{m} \sigma^2(p_i) \sum_{k=1}^{n} Q_{jk}^{-1} a_{ik} a_{ik},$$

(10)

where $\sigma^2(p_i)$ and $\sigma^2(x_j)$ are the variance of $p_i$ and $x_j$, respectively. In order to make a fair noise behavior comparison between the two systems, we adopt the same assumption made for the line-integral system in Huesman’s work, namely, that the uncertainties of all measured integrals ($p_i$) are equal, thus, the variance of the projection data $\sigma^2(p_i)$ is equal to a constant $\sigma_p^2$, and substituting Eq. (4) into Eq. (10) gives

$$\sigma^2(x_j) = \frac{D}{m \times dp^5} \sigma_p^2 Q^{-1}_{jj}.$$  

(11)

As matrix $Q$ is independent of the acquisition geometry (e.g., the number of projections), $Q^{-1}$ is also independent when $Q$ is invertible. When a full dataset is acquired, matrix $Q$ is a full rank matrix that guarantees the invertibility of $Q$. Equation (11) assumes that all measured planar integrals are equally uncertain, which is the same as the assumption made in Huesman’s work. In practice, the shape of the object is arbitrary such that the variance of the projection data $p_i$ varies from case to case. Here $p$ can be considered as a
weighted average of the planar-integrals $p_i$. Since $dp$ is generally a small value and $a_i a_{ij} = dp^2$ if nonzero, the average value of the planar integrals can be expected to be a rough estimation of $p$. Therefore, assuming Poisson noise, $Q_{jj}^p$ is approximately equal to the variance of the average value of the planar integrals that is equal to $N_p/m$, where $N_p$ is the total counts received. Nevertheless, for cases to which the above estimation does not apply, practical assumptions need to be made.

### C. A sufficient number of projections

In Huesman’s work,\(^5\) it was shown that for an infinite number of line-integral projections, the counterparts of matrices $Q$ and $Q^{-1}$ (denoted as $M$ and $M^{-1}$ in Ref. 6, respectively) were very reliable for arbitrary values of $D/dp$, and the diagonal elements of $M^{-1}$ are found to be equal to 1.59. In addition, Huesman showed that the average value of the diagonal elements of $M^{-1}$ converged to 1.59 when the value of $D/dp$ was greater than 1.5. In this section, we investigate $Q_{jj}^{-1}$ similarly for the planar-integral system using a sufficient number of data points.

Matrices $Q$ are generated for five uniform spheres with increased $D/dp$ (the number of voxels across the diameter, equal to 6, 8, 10, 12, and 14) using two acquisition setups. In order to obtain a sufficiently large number of data points, the number of projection angles ($m_\phi$ and $m_\theta$) and the number of detector bins ($m_x$) are set to be large enough for the two acquisitions. The numbers of projections $m = m_\phi \times m_\theta \times m_x$ used in the two acquisition setups are $140 \times 140 \times 140$ ($m_\phi = 140$, $m_\theta = 140$, and $m_x = 140$) and $280 \times 280 \times 280$ ($m_\phi = 280$, $m_\theta = 280$, and $m_x = 280$), respectively.

The first setup makes $(m_x \times dp)/D \geq 10$, $(m_\phi \times dp)/D \geq 10$, and the second one makes $(m_\phi \times dp)/D \geq 10$, $(m_\theta \times dp)/D \geq 10$, $(m_\phi \times dp)/D \geq 10$, and $(m_\theta \times dp)/D \geq 10$. The detector is set to be slightly larger than $D \times D$, and both the spinning angles ($\theta$) and rotation angles ($\phi$) are uniformly distributed over $180^\circ$. The $(i,j)$th element $Q_{ij}$ of each matrix $Q$ is calculated by the intersection area of the $i$th projection plane and the $j$th voxel of the image. The diagonal elements $Q_{jj}$ for each $D/dp$ case are found to be approximately equal to a constant, which is shown in Table I as $Q_{jj}^p$. Matrices $Q$ are inverted for all cases to obtain the corresponding inverse matrices $Q^{-1}$. The diagonal elements $Q_{jj}^{-1}$ of each matrix $Q^{-1}$ are found to be quite stable for all $j$’s as expected. The average values of the $Q_{jj}^{-1}$ are shown in Table I, and denoted as $(Q_{jj}^{-1})_{avg}$. The values of $(Q_{jj}^{-1})_{avg}$ are calculated using all the diagonal elements except those at the edge of the reconstruction region.

It is shown that when $m$ is sufficiently large, both $Q_{jj}$ and $(Q_{jj}^{-1})_{avg}$ remain stable for different values of $D/dp$ and $m$. The variances of $Q_{jj}^{-1}$ for all $D/dp$ cases in Table I are found approximately equal to 0.05. The approximate values $(Q_{jj}^{-1})_{avg}$ shown in Table I are about equal to 3.4.

In general, the signal-to-noise ratio (SNR) is defined as the ratio of the average counts of the activities to the standard deviation.\(^10\) Assuming a Poisson noise model, the SNR of the planar-integral projection data $(SNR_p)$ is defined as

\[
SNR_p = \left( \frac{\langle \rho \rangle}{\sigma_p} \right) = \sqrt{\frac{N_p}{\langle \rho \rangle m}} = \sqrt{\frac{N_p}{\langle \rho \rangle m}},
\]

where $\langle \rho \rangle$ denotes the average counts measured at each detector bin and $N_p$ represents the total counts measured by the detector over all projection angles. Since the average density of the reconstructed image $\langle \chi \rangle$ and the average count received at each detector bin $\langle \rho \rangle$ have the following relationship according to the acquisition geometry:

\[
\langle \rho \rangle = \frac{4}{3} \pi \left( \frac{D}{2} \right)^3 \langle \chi \rangle = \frac{\pi D^2}{6} \langle \chi \rangle.
\]

The division of Eq. (13) by the square root of Eq. (11) gives

\[
\frac{\langle \rho \rangle}{\sigma_p} = \frac{6}{\pi D^2} \left( \frac{Q_{jj}^{-1} m \times dp^5}{D} \right)^{1/2} \langle \rho \rangle
\]

\[
= \frac{6}{\pi} \left( \frac{m \times dp^5}{Q_{jj}^{-1} D^5} \right)^{1/2} \langle \rho \rangle
\]

\[
= \frac{6}{\pi} \left( \frac{m \times dp^5}{Q_{jj}^{-1} D^5} \right)^{1/2} \langle \rho \rangle
\]

Substituting $Q_{jj}^{-1}$ with 3.4 in Eq. (14) and assuming $dp = 1$ gives

\[
SNR_x = \frac{6}{\pi} \left( \frac{m}{\sqrt{3.4 D^3}} \right) \frac{1}{P_{\rho}}
\]

\[
= 1.04 \left( \frac{m}{D^3} \right)^{1/2} \frac{1}{P_{\rho}}
\]

Referring to the similar result for line-integral projection data presented in Huesman’s work,\(^5\) the division of the SNR of the image reconstructed using planar-integral data $(SNR_p)$ by the SNR $(SNR_{p\rho})$ of the image reconstructed using line-integral data can be expressed as...
with Eq.

\[ \text{SNR}_I \left( \frac{m}{D^2} \right)^{1/2} \text{SNR}_{lp} = \left( \frac{m}{D^2} \right)^{1/2} \sqrt{\frac{N_p}{m}} \left( \frac{I}{D^2} \right)^{1/2} \sqrt{\frac{N_p}{m}} = \sqrt{\frac{N_p}{N_p D^2}}, \]

(16)

D. An insufficient number of projection angles

As mentioned above, to guarantee the stability of \((Q^{-1}_{ij})\) over all j’s, a sufficiently large number of projections are needed. When \(m_\theta\) is sufficiently large, but \(m_\phi\) or \(m_\phi^*\) is comparable to \(D/dp\), the values of \((Q^{-1}_{ij})\) are not stable over all j’s, and the average value \((Q^{-1}_{ij})_{\text{avg}}\) varies with the number of projection angles. In order to investigate the effects of an insufficient number of projection angles, \(m_\theta\) is assumed to be sufficiently large, while \(m_\theta\) and \(m_\phi^*\) are relatively small. First, \(Q\) matrices are generated for two \(D/dp\) cases \((D/dp=10\) and 14) with a sufficiently large \(m_\theta\) and with the ratio of the number of rotation angles \(m_\theta\) to \(D/dp\) \((m_\theta \times D/dp)\) equal to 0.75, 1.0, 1.25, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, and 5.0, respectively. The average value of \(Q^{-1}_{ij}\) for each case is calculated using all elements except those at the edge of the object. The resulting average values \((Q^{-1}_{ij})_{\text{avg}}\) are shown in Table II. Here, \(m_\theta\) and \(m_\phi^*\) are both set to 140, which is sufficiently large, as verified in the previous section. Angles \(\theta\) and \(\phi\) are uniformly distributed over 180°.

Matrices \(Q\) are then regenerated for the two \(D/dp\) cases, but with a sufficiently large \(m_\theta\) and with the ratio of the number of spinning angles \(m_\phi^*\) to \(D/dp\) \((m_\phi^* \times D/dp)\) equal to 0.75, 1.0, 1.25, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, and 5.0, respectively, and, this time, \(m_\theta\) and \(m_\phi^*\) set to 140. The resulting average values \((Q^{-1}_{ij})_{\text{avg}}\) are shown in Table III.

Note that both \(m_\theta\) and \(m_\phi^*\) are integers. The actual \(m_\theta\) and \(m_\phi^*\) used to generate the data in Table II and Table III are obtained by rounding the values of \((m_\theta \times D/dp) / (D/dp)\) and \((m_\phi^* \times D/dp) / (D/dp)\) to the nearest integers. As can be seen in Table II and Table III, the values of \((Q^{-1}_{ij})_{\text{avg}}\) decrease as the number of angles increases and approach their corresponding values shown in Table I as expected. The converged \((Q^{-1}_{ij})_{\text{avg}}\) are quite stable for different \(D/dp\) cases. Table II and Table III show that when the number of rotation angles \(m_\theta\) reaches approximately 2.0 times \(D/dp\) \([m_\theta = 2.0(D/dp)]\), and the number of spinning angles \(m_\phi^*\) reaches about 2.5 times \(D/dp\) \([m_\phi^* = 2.5(D/dp)]\), \((Q^{-1}_{ij})_{\text{avg}}\) are very close to their counterparts for which a sufficiently large num-

<table>
<thead>
<tr>
<th>(Q^{-1}<em>{ij})</em>{\text{avg}}</th>
<th>0.75</th>
<th>1.0</th>
<th>1.25</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_\theta \times D/dp)</td>
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</tr>
</tbody>
</table>

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number of angles are applied. In addition to the average values \((Q_{ij}^{-1})_{\text{avg}}\), the variances of the \(Q_{ij}^{-1}\) over all \(j\)'s are also found to decrease as \(m_\phi\) and \(m_\theta\) increase as well, and become stable for \(m_\phi\geq 2.0(D/dp)\) and \(m_\theta\geq 2.5(D/dp)\) (\(\approx 0.05\)). In Table II and Table III, it can be seen that for \(m_\phi\times dp/D=1.0\) and \(m_\theta\times dp/D=1.0\), \((Q_{ij}^{-1})_{\text{avg}}\) is much greater than 3.4, which suggests that setting the numbers of rotation angles and spinning angles to be equal to the number of voxels across the object diameter \((D/dp)\) is not proper for 3D reconstruction using planar-integral data.

E. An insufficient number of detector samplings

In this section, the effects of an insufficient number of detector samplings are investigated using similar methods. The average values of \((Q_{ij}^{-1})\) are calculated for several values of \(\Delta z/dp\) \((\Delta z/dp=0.1, 0.2, 0.3, 0.5, 0.7, \text{and} 1.0)\). Two \(D/dp\) cases \((D/dp=10 \text{ and } 14)\) are studied. The number of projection angles are sufficiently large \((m_\phi=m_\theta=140)\). Table IV shows the obtained \((Q_{ij}^{-1})_{\text{avg}}\).

It has been shown that the values of \((Q_{ij}^{-1})_{\text{avg}}\) increase substantially for values of \(\Delta z/dp\) greater than 0.5, but they are quite stable for \(\Delta z/dp\) less than 0.3. It suggests that to reconstruct an image with voxel dimension \(dp\), the sampling interval between planar integrals (or the bin size) has to be considerably less than \(dp\). In order to make efficient use of the data, a sampling interval of less than \(0.5dp\) is desired. This is consistent with the Huesman’s result, which suggested that for 2D reconstruction with line-integral projection data, the sampling interval required is less than \(0.7dp\).

F. Computer simulations

In this section, noise propagation behaviors of the planar-integral system and the line-integral system are compared based on SNRs using computer simulations. To simplify the simulation, a sphere of uniformly distributed radioactivity with a diameter \(D\) is the 3-D object to be reconstructed using the planar-integral system, while a uniform disk with the same diameter is the 2-D object for the line-integral system. The voxel dimension and the slice thickness are both equal to \(dp\).

The ML-EM and OS-EM algorithms are the most widely used iterative algorithms in SPECT. They have many desirable features, such as the non-negative property of the their solutions. The ML-EM algorithm suffers slow convergence, which is a big problem, in practice. The ordered-subset EM (OS-EM) algorithm accelerates the convergence successfully by breaking up the full set of projection data into a series of mutually exclusive subsets and applying the reconstruction algorithm to each subset sequentially. Both the ML-EM and the OS-EM algorithms are constructed based on the maximum likelihood criteria.8 They provide maximum likelihood solutions to the system equation (1).

Other approaches to solve Eq. (1) with a better convergence rate than the OS-EM algorithm are the conjugate gradient minimum residual (CG-MR) approaches, which are constructed based on the least-squares criteria. They provide least-squares solutions by solving the asymmetric system equations (19) generalized from (1):

\[
C^T A x = C^T p, \tag{19}
\]

where \(p\) is the measurement vector, \(A\) is the system matrix, and \(C\) is a preconditioning operator.17 Matrix \(C\) may not be the same as \(A\). To increase the convergence rate, matrix \(C\) is usually chosen to make matrix \(C^T A\) an approximate generalized inverse of \(A\) such that \(C^T A\) is close to the identity matrix. One efficient way to design the preconditioning operator \(C\) is to use a filtered-backprojection (FBP) type operator.

In our simulation, we adopted the conjugate gradient residual minimization algorithm (CG-MR) employed in Refs. 4 and 16 with a FBP preconditioning operator. For the line-integral system, projection data were filtered with a ramp filter before backprojection, while for the planar-integral system, the data were filtered with a second-derivative filter.4 The reason that we choose a CG-MR algorithm instead of the OS-EM in the simulation is not only because of the better convergence rate of the CG-MR, which is a desired feature for the reconstruction of the planar-integral system, but also because both Huesman’s and our own noise behavior analyses are based on a least-square reconstruction, which can be simulated using the CG-MR but not the OS-EM algorithm.

Here we also adopt the assumption that a Poisson noise model and a Gaussian noise model are reasonably identical. This assumption is realistic, especially for the slat collimation case due to its high geometric efficiency.

G. Noise propagation stability

The noise response is always the most important determinant of image quality. The noise behaviors of the two systems have been thoroughly discussed in Refs. 5 and 7 for analytical methods, but not for iterative algorithms. In the previous sections, we have found that for a sufficient number of projection angles and detector samplings (e.g., \(m_\phi\geq 2.0 \times D/dp, m_\theta\geq 2.5 \times D/dp, \text{and } \Delta z \leq 0.5dp\)), the variance of the reconstructed image can be quite stable for a least-squares reconstruction. It has been shown that for line-integral systems, both the mean and the variance of the reconstructed 2D image can be quite stable after a number of iterations when a halved sampling interval and a sufficiently large number of projection angles are applied. Therefore, our simulations are focused on the planar-integral system and expected to obtain a similar result. Two uniform spheres (diameter \(D=12dp\) and \(40dp\)) are presented as the radioactive objects in the simulations. The distribution density \(p\) of each

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**Table IV.** Average values of \((Q_{ij}^{-1})_{\text{avg}}\) for an insufficiently large number of detector samplings \(m_\phi, m_\theta\).

<table>
<thead>
<tr>
<th>(D/dp)</th>
<th>(\Delta z/dp)</th>
<th>(\Delta z/dp)</th>
<th>(\Delta z/dp)</th>
<th>(\Delta z/dp)</th>
<th>(\Delta z/dp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_{ij}^{-1})_{avg}</td>
<td>10</td>
<td>3.405</td>
<td>3.406</td>
<td>3.411</td>
<td>3.434</td>
</tr>
</tbody>
</table>
sphere is arbitrarily set to 120. The noise free planar-projection data are generated based on the symmetric geometry of the sphere:

$$p_i = \rho \left[ \left( \frac{D}{2} \right)^2 - t_i^2 \right],$$

where $p_i$ represents the $i$th noise free planar-projection datum, $D$ is the diameter of the sphere, and $t_i$ denotes the distance between the $i$th projection plane and the center of the sphere. The noisy projection data $\tilde{p}_i$ are generated by adding Poisson noise to the noise free projection data $p_i$. A 3D Conjugate Gradient Minimal Residual (CG-MR) algorithm is implemented with $m_{\phi}=2.0 \times D/dp$, $m_{\Delta}=2.5 \times D/dp$, and $\Delta z=0.5 dp$. The SNR values are calculated within the spheres excluding the voxels at the edge. Figure 5 shows that the planar-projection system can also achieve good noise propa-

![Fig. 5. Noise propagation stability of the CG-MR algorithm using planar integral data (a) $D=12 dp$; (b) $D=40 dp$.](image-url)
position of each slice. The SNR of each 2D reconstructed slice is calculated within the center region of the slice excluding those voxels at the edge and compared with the SNR of a corresponding sphere reconstructed using planar-projection data. For both systems, the same arbitrary statistics (\(\rho = 120\)) are initialized. The geometric efficiency factor \(F\) is arbitrarily chosen to be \(F = 15\). No additional smoothing is applied. All SNRs are calculated after the reconstructions iterate to their noise stable state.

Figure 6 shows that for a certain statistic (isotope density \(\rho = 120\)) and a fixed geometric factor \(F = 15\), the SNR comparison ratio of the two systems decreases as the size of the sphere increases. The four dash-dotted lines (a, b, c, and d) shown in Fig. 6 are the \(F/\rho\) model fitted curves with \(c = 1.1, 1.0, 0.9,\) and \(0.8\), respectively. One can see that the relationship between the SNR comparison ratio and the size of the object \(D\) is approximately in the form \(F/\rho\), and, furthermore, the constant \(c\) for the curve of SNR/\(\rho\) versus the diameters \(D/dp\) is approximately equal to 0.95, which is very close to 0.82 given in Eq. (18). The discrepancy might be caused by the assumption of the equal uncertainty of the projection data and the approximation of Poisson noise model for a high-count least-square reconstruction.

The reconstructed SNR comparison ratios with different statistics (different \(\rho\) values) are shown in Fig. 7. Note that the isotope distribution density \(\rho\) is proportional to the total number of projection counts received by the detector. Large \(\rho\) values represent high-count statistics, while small \(\rho\) values indicate low-count statistics. In Fig. 7, a fixed efficiency parameter \(F = 15\) and the same scanning periods are assumed for both systems. Two spheres with diameter \(D = 20dp\) and \(D = 40dp\) are investigated. It can be seen that the ratios remain constant for different statistics.

H. SNR comparison between the two systems

In this section, we provide a simulation to verify the relationship shown in (18) using the CG-MR algorithm. Since the noise propagation of the two systems can be stable after a certain number of iterations, it is possible to compare the two systems independently of iteration numbers. For the planar-integral system, the simulation setups are similar to those mentioned in Sec. II G, but spheres with several diameter values \(D\) are studied (\(D = 8dp, 10dp, 20dp, 30dp, 40dp, 50dp,\) and \(60dp\)). The SNR of each reconstructed sphere is calculated within the center region of the sphere except the edge voxels. The center slice (slice thickness is equal to \(dp\)) of each sphere is also reconstructed using a 2D CG-MR algorithm with line-integral data. The noise free line-projection data are generated similarly, according to the symmetric geometry of the disk:

\[
L_i = 2\rho \sqrt{\left(\frac{D}{2}\right)^2 - t_i^2},
\]

where \(L_i\) is the \(i\)th noise free line-projection datum, and here \(t_i\) represents the distance between the center of the disk and the \(i\)th line-projection path. Poisson noise is added to each \(L_i\) to generate noisy projection data \(\hat{L}_i\) for the reconstructions. There are \(1.5D/dp\) projection angles applying to data acquisition of each slice. The SNR of each 2D reconstructed slice is calculated within the center region of the slice excluding those voxels at the edge and compared with the SNR of a corresponding sphere reconstructed using planar-projection data. For both systems, the same arbitrary statistics (\(\rho = 120\)) are initialized. The geometric efficiency factor \(F\) is arbitrarily chosen to be \(F = 15\). No additional smoothing is applied. All SNRs are calculated after the reconstructions iterate to their noise stable state.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{A noise propagation comparison between the planar-integral system and the line-integral system (SNR/\(\rho\)) based on the radius of uniform spheres using CG-MR algorithms (\(\rho = 120\)). Fitted curves with \(c = 1.1, 1.0, 0.9,\) and \(0.8\) are denoted as a, b, c, and d, respectively.}
\end{figure}
The efficiency factor $F$ is a factor that depends on both the detector-to-object distance and the detector sizes of the two systems. In Lodge’s work,\textsuperscript{5} $F$ was found to vary from 12 to 28 when the same detector size was applied to both systems. $F=15$ was arbitrarily chosen in Lodge’s simulation study. For direct comparison with Lodge’s work, we also employed $F=15$ in the above simulation, assuming that the two systems have the same detector size. When the narrow strip detector is much smaller than the detector of the line-integral system, $F$ might consequently be less than 15. For instance, when the width of the strip detector is halved, compared to the width of the detector of the line-integral system, $F=7$ or 8 would be an expected estimate. To verify that our results obtained above also stand for various values of $F$, we resimulated $\text{SNR}_x/\text{SNR}_{nl}$ with $F=8$ using spheres of $D=8$, 20, 40, and 60. The curve obtained of $\text{SNR}_x/\text{SNR}_{nl}$ is shown in Fig. 8, and the constant $C$ is roughly equal to 0.95, as expected.

![Fig. 7. The noise propagation comparison between the planar-integral system and the line-integral system ($\text{SNR}_x/\text{SNR}_{nl}$) versus different noise statistics ($\rho=10, 30, 60, 120, \text{and 240}$) ($F=15$) ($D=20/dp$ and $40/dp$).](image)

![Fig. 8. A similar SNR$_x$/SNR$_{nl}$ simulation to that shown in Fig. 6 with $F=8$ and $D=8, 20, 40, \text{and 60}$. Fitted curves with $c=1.1, 1.0, 0.9, \text{and 0.8 are denoted as a, b, c, and d, respectively.}](image)
III. CONCLUSIONS AND FUTURE WORK

In this work, we provide an evaluation of the noise propagation behavior of the planar-integral system (e.g., a rotating slat system) and the line-integral system (e.g., a parallel-hole system) based on a least-square iterative reconstruction. Our analysis is consistent with Lodge’s work, which was for an FBP based reconstruction. We also provide the desired data sampling and projection angle conditions for the planar-integral system for efficient data acquisition, which is an extension of Huesman’s work on the line-integral system.6

The noise propagation features of iterative reconstructions using planar-integral data are dependent on the number of projection angles \( m_\theta \) and \( m_\phi \) and the number of detector samplings \( m_z \). Equation (11) expresses the relationship between the variance of the least-squares solution of Eq. (1) and the variance of the measured planar integrals. When \( m_\theta \), \( m_\phi \), and \( m_z \) are sufficiently large compared to the number of voxels across the diameter of the object (\( D/dp \)), the diagonal elements \( Q_{ij}^{-1} \) of the inverted matrix \( Q^{-1} \) turn out to be fairly stable for arbitrary values of \( D/dp \). The approximate value of \( Q_{ij}^{-1} \) is found to be roughly equal to 3.4. Assuming Poisson noise, the relationship between the SNR of the solution and the SNR of its planar integrals is given in Eq. (15). Comparing Eq. (15) with the corresponding relationship for the line-integral case obtained in Huesman’s work, the SNR ratio of the solutions of the two systems is expressed in Eq. (18). This SNR ratio shows the tradeoffs between planar-integral and line-integral projections as a function of the size of the object in the form of \( 1/\sqrt{D} \). This indicates that the large increase in geometry efficiency afforded by using planar-integral projections does not guarantee a significant reduction in reconstructed image noise. For a fixed resolution \( dp \), the planar projection method demonstrated no advantage for large objects. For smaller objects, however, the planar projection method may outperform the line projection method depending on a geometric efficiency factor \( F \). The resolution comparison of the two systems is not investigated in this work, but it has been discussed in Refs. 7 and 22 that the rotating-slat system can achieve slightly better resolution than the conventional parallel-hole system.

For an insufficient number of projection angles (either \( m_\theta \) or \( m_\phi \) is comparable to \( D/dp \), and \( m_z \) is sufficiently large), the average values \( (Q_{ij}^{-1})_{avg} \) of \( Q_{ij}^{-1} \) are calculated for several values of \( m_\theta \) and \( m_\phi \). It is shown that \( (Q_{ij}^{-1})_{avg} \) decreases with an increasing number of angles. When \( m_\theta \) reaches approximately 2.0 times \( D/dp \), and \( m_\phi \) reaches about 2.5 times \( D/dp \), \( (Q_{ij}^{-1})_{avg} \) is very close to the approximate value \( Q_{ij}^{-1} \) obtained when the number of projection angles is sufficiently large. For an insufficient number of planar integrals for each angle (both \( m_\theta \) and \( m_\phi \) are large enough, but \( m_z \) is comparable to \( D/dp \)), the average values \( (Q_{ij}^{-1})_{avg} \) increase considerably for values of \( \Delta z/dp \) greater than 0.5, but they remain quite stable for \( \Delta z/dp \) less than 0.3. It suggests that to efficiently utilize the data, the sampling interval between planar integrals (or the bin size) must be less than 0.5\( dp \). The noise propagation stability of the small detector sampling integral (\( \Delta z=0.5dp \)) and the relationship given in Eq. (18) are verified using computer simulations.

The SNR comparison study in this work is based on the assumption that the variance of the planar integrals is a constant, which is similar to the assumption made in Huesman’s work.6 The constant variance is approximately estimated as the variance of the average value of the measured planar integrals. Therefore, it provides a rough evaluation of the noise behaviors of the planar and line projection methods. For a more accurate investigation, practical assumptions need to be made.

In addition, and to be consistent with the work of Huesman and Lodge, we assumed that the projection data were generated using 3D radon transforms for the rotating-slat system and 2D radon transforms for the parallel-hole system. The variation of the sensitivity within the FOV and detector blurring effects (solid angle effects) were not considered.23 But in practice, these factors can affect the statistical nature of the projection data as well as the noise propagation in the reconstruction, and should be evaluated in future work. First, the sensitivity of the rotating slat system decreases as the object-to-detector distance increases, while the sensitivity of the parallel-hole system remains constant in the FOV. The rotating-slat system exhibits a 1/r sensitivity dependence (where \( r \) is the object-to-detector distance) resulting in better performance as the object is put closer to the detector.22 This effect could be modeled by a parameter that is dependent on the object-to-detector distance as well as the configuration of the slats, and would play a similar role to the geometry efficiency parameter \( F \). Second, due to the finite gap size of the rotating-slat collimator and the finite hole size of the parallel-hole collimator, the blurring effects of the rotating-slat system and the parallel-hole system vary with different collimator configurations. Therefore, when the detector response is modeled, the noise propagation of the two systems should be compared based on reconstructed images that have the same spatial resolution.

7M. A. Lodge, S. Webb, M. A. Flower, and D. M. Binnie, “The experi-


