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Exact fan-beam and $4\pi$-acquisition cone-beam SPECT algorithms with uniform attenuation correction

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This paper presents analytical fan-beam and cone-beam reconstruction algorithms that compensate for uniform attenuation in single photon emission computed tomography. First, a fan-beam algorithm is developed by obtaining a relationship between the two-dimensional (2D) Fourier transform of parallel-beam projections and fan-beam projections. Using this relationship, 2D Fourier transforms of equivalent parallel-beam projection data are obtained from the fan-beam projection data. Then a quasi-optimal analytical reconstruction algorithm for uniformly attenuated Radon data, developed by Metz and Pan, is used to reconstruct the image. A cone-beam algorithm is developed by extending the fan-beam algorithm to $4\pi$ solid angle geometry. The cone-beam algorithm is also an exact algorithm. © 2005 American Association of Physicists in Medicine. [DOI: 10.1118/1.2068907]

Key words: Fan-beam, cone-beam, attenuation correction

I. INTRODUCTION

The use of converging beam geometries has been investigated for several single photon emission computed tomography (SPECT) applications, such as brain, cardiac, and small animal imaging.$^{1,2}$ Pinhole collimation and multi-pinhole collimation are examples of cone-beam imaging geometries and are often used in small animal SPECT imaging.$^{3-6}$ The cone-beam reconstruction problem alone presents a significant challenge; however, the attenuation presents an even greater challenge for reconstructing cone-beam data. Some researchers have investigated the cone-beam reconstruction problems with the cone-beam focal point distributed on a sphere.$^{7,8}$ The goal of this paper is to develop an exact analytical cone-beam algorithm that compensates for uniform attenuation with the cone-beam focal point distributed on a sphere.

There are many analytical parallel-beam reconstruction algorithms$^{9-16}$ that correct for uniform attenuation. In this paper we first consider the imaging geometry where the fan-beam collimator rotates in a circle around the object. Other published research$^{17,18}$ extends the results for a uniform attenuator from parallel-beam geometry to fan-beam geometry. Those analytical fan-beam algorithms are basically the fan-beam versions of the Tretiak-Metz algorithm.$^{10}$

Metz-Pan’s quasi-optimal parallel-beam method$^{16,19}$ has demonstrated superior noise properties; this paper extends their parallel-beam quasi-optimal algorithm to fan-beam and cone-beam algorithms. In our fan-beam algorithm, the two-dimensional (2D) Fourier transform of modified projection data is obtained first, then a filtering procedure is performed. Next, an inverse fast Fourier transform is performed to obtain the reconstructed image. This fan-beam algorithm is then extended to the cone-beam $4\pi$ imaging geometry. For a given normal direction the cone-beam data are processed slice-by-slice using the new fan-beam algorithm. A summation of the reconstructions in all normal directions is calculated as the final image.

The proposed cone-beam algorithm is an exact extension of the new fan-beam algorithm.

II. FAN-BEAM ALGORITHM

Let the parallel projection data of a 2D object with a uniform attenuator be $p(t, \theta)$ and the fan-beam projection data of the same object with the same attenuator be $g(\sigma, \beta)$. The modified attenuated sinogram of parallel projection data is defined as (i.e., the exponential Radon transform) $m(t, \theta)$:

$$m(t, \theta) = e^{\mu D(t, \theta)}p(t, \theta),$$  

(1)

where $D(t, \theta)$ is the distance as shown in Fig. 1.

The modified attenuated sinogram of fan-beam projection data is defined as $n(\sigma, \beta)$:

$$n(\sigma, \beta) = e^{\mu D(\sigma, \beta)}g(\sigma, \beta),$$  

(2)

where $D(\sigma, \beta)$ is the distance as shown in Fig. 1. Variables $t$, $\theta$, $\sigma$, and $\beta$ are described in Fig. 1. An arbitrary projection ray can be expressed either in the parallel-beam or fan-beam notation, that is, $m(t, \theta) = n(\sigma, \beta)$, with the following transformation of variables $t$, $\theta$, $\sigma$, and $\beta$:  

Changing the order of integration yields

\[ M_k(\omega) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} n(\sigma, \beta) d\beta d\sigma \]

\[ \times \int_0^{2\pi} e^{i2\pi\omega t} e^{-ik\theta} d\theta dt \]

\[ \times \delta(-\theta + \sigma + \beta)e^{-i\omega t}\delta(t)dt. \]

(9)

The delta function has the property \(^{20}\)

\[ \delta(g(x)) = \sum_{a, g(a) = 0} \frac{\delta(x-a)}{|g'(a)|}. \]

(10)

Let

\[ g(t) = \sigma - \sin^{-1}\left(\frac{t}{R}\right) = 0, \]

(11)

then

\[ g'(t) = \left(\sigma - \sin^{-1}\left(\frac{t}{R}\right)\right)' = -\frac{1}{\sqrt{1 - \left(\frac{t}{R}\right)^2}}. \]

(12)

Because \(-\pi/2 \leq \sigma \leq \pi/2\), solving Eq. (11) yields

\[ t = R \sin \sigma. \]

(13)

Substituting Eq. (13) into Eq. (12) yields

\[ g'(t) = -\frac{1}{\cos \sigma}. \]

(14)

From Eqs. (10), (13), and (14) we can obtain

\[ \delta(g(t)) = \delta\left(\sigma - \sin^{-1}\left(\frac{t}{R}\right)\right) = |\cos \sigma| \delta(t - R \sin \sigma) \]

\[ = \cos \sigma \delta(t - R \sin \sigma), \]

(15)

and

\[ \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \delta\left(\sigma - \sin^{-1}\left(\frac{t}{R}\right)\right) \delta(-\theta + \sigma + \beta)e^{-i2\pi\omega t} e^{-ik\theta} d\theta dt \]

\[ = e^{-i2\pi\omega R} \sin \sigma e^{-i(k(\sigma + \beta))} \cos \sigma. \]

(16)

Then Eq. (9) can be simplified as

\[ M_k(\omega) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} n(\sigma, \beta)e^{-i2\pi\omega R} \sin \sigma e^{-i(k(\sigma + \beta))} \]

\[ \times \cos(\sigma) d\beta d\sigma. \]

(17)

\[ t = R \sin \sigma, \]

(3a)

\[ \theta = \sigma + \beta, \]

(3b)

where \( R \) is focal length of the fan-beam geometry, and

\[ 0 \leq \theta \leq 2\pi, \]

(4a)

\[ 0 \leq \beta \leq 2\pi, \]

(4b)

\[ -\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2}. \]

(4c)

From Eqs. (3) and (4), we have

\[ \sigma = -\sin^{-1}\left(\frac{t}{R}\right), \]

(5a)

\[ \beta = \theta - \sigma. \]

(5b)

Using the property of the delta function we have

\[ m(t, \theta) = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} n(\sigma, \beta) \delta\left(\sigma - \sin^{-1}\left(\frac{t}{R}\right)\right) \]

\[ \times \delta(\beta - (\theta - \sigma)) d\beta d\sigma. \]

(6)

Performing the 2D Fourier transform of \( m(t, \theta) \) yields

\[ M_k(\omega) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} m(t, \theta) e^{-i2\pi\omega t} e^{-ik\theta} d\theta dt. \]

(7)

Substituting Eq. (6) into Eq. (7) yields

\[ M_k(\omega) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} e^{-i2\pi\omega t} e^{-ik\theta} d\theta dt \]

\[ \times \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} n(\sigma, \beta) \delta\left(\sigma - \sin^{-1}\left(\frac{t}{R}\right)\right) \delta(-\theta + \sigma + \beta) d\beta d\sigma. \]

(8)

Changing the order of integration yields

Evaluating the integration with respect to \( \beta \) gives
\[ M_k(\omega) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-i2\pi R \sin(\sigma) - ik\sigma} \cos(\sigma) d\sigma \times \int_0^{2\pi} n(\sigma, \beta) e^{-ik\beta} d\beta \]
\[ = \int_{-\pi/2}^{\pi/2} N_k(\sigma) \cos(\sigma) e^{-i2\pi R \sin(\sigma) + k\sigma} d\sigma, \quad (18) \]

where
\[ N_k(\sigma) = \frac{1}{2\pi} \int_0^{2\pi} n(\sigma, \beta) e^{-ik\beta} d\beta. \quad (19) \]

Integrand \( n(\sigma, \beta) \) in Eq. (18) is the modified attenuated sinogram of the fan-beam projection data, while integrand \( m(t, \theta) \) in Eq. (7) is the modified attenuated sinogram of the equivalent parallel-beam projection data. Function \( M_k(\omega) \) is defined as the 2D Fourier transform of the modified attenuated sinogram of the equivalent parallel-beam projection data. Identity (18) gives a method to obtain the 2D Fourier transform of the modified attenuated sinogram of the equivalent parallel-beam projection data from the modified attenuated sinogram of the fan-beam projection data. Comparing Eq. (7) with Eq. (18), the exponential factor is changed from the normal Fourier transform factor \( e^{-i2\pi \omega t} \) to a more general factor \( e^{-i2\pi R \sin(\sigma) + k\sigma} \), which is not only related to the frequency \( \omega \), but is also related to the focal length \( R \) and expansion order \( k \).

Our new fan-beam algorithm is based on the quasi-optimal algorithm for the parallel-beam exponential Radon transform presented by Metz and Pan.\(^{16}\) The algorithm is described briefly as follows.

Let the 2D activity distribution function in polar coordinates be \( a(r, \phi) \), the attenuated parallel sinogram be \( p(t, \theta) \), and the modified attenuated sinogram (i.e., the exponential Radon transform) be \( m(t, \theta) \). Performing the Hankel transform and Fourier transform of the activity \( a(r, \phi) \) gives
\[ A_k(\omega) = (-i)^k \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} a(r, \phi) e^{-ik\phi} J_k(2\pi\omega r) r dr d\phi, \quad (20) \]
where \( J_k(2\pi\omega r) \) is the \( k \)-th order Bessel function of the first kind and \( k = 0, \pm 1, \pm 2, \ldots \).

Performing the 2D Fourier transform of the modified attenuated sinogram \( m(t, \theta) \) yields
\[ M_k(\omega) = \frac{1}{2\pi} \int_{\theta=0}^{\pi} \int_{r=\infty}^{\infty} m(t, \theta) e^{-ik\theta} e^{-i2\pi \omega r} dtd\theta, \quad (21) \]
with \( k = 0, \pm 1, \pm 2, \ldots \).

From Metz and Pan’s paper\(^{16}\)
\[ A_k(\omega) = \lambda [\gamma(\omega_\mu)]^k M_k(\omega_\mu) + (1 - \lambda) [\gamma(-\omega_\mu)]^k M_k(-\omega_\mu), \quad (22) \]
where
\[ \omega_\mu = \sqrt{\omega^2 + (\mu/2\pi)^2}, \quad (23a) \]
\[ \gamma(\omega_\mu) = \frac{\sqrt{\omega^2 + (\mu/2\pi)^2}}{\omega_\mu + \mu/2\pi}, \quad (23b) \]
\[ 0 \leq \lambda \leq 1. \quad (23c) \]

The proposed fan-beam algorithm consists of the following steps.

1. Modify each attenuated fan-beam line-integral \( q(\sigma, \beta) \) by a scaling factor \( e^{\mu D(\sigma, \beta)} \) obtaining \( n(\sigma, \beta) = e^{\mu D(\sigma, \beta)} q(\sigma, \beta) \), where \( \mu \) is the linear attenuation coefficient and \( D(\sigma, \beta) \) is the distance defined in Fig. 1.
2. Use Eq. (18) to obtain the 2D Fourier transform of the equivalent parallel-beam projection data \( M_k(\omega) \).
3. Use Eq. (22) to perform frequency shifting and weighting and to obtain the 2D Fourier transform of the 2D activity distribution function \( A_k(\omega) \).
4. Perform an inverse Fourier transform of \( A_k(\omega) \) with respect to \( k \), obtaining
\[ A(\omega, \phi) = \sum_{k=-\infty}^{\infty} A_k(\omega) e^{ik\phi} \quad (24) \]
in polar coordinates.
5. Express \( A(\omega, \phi) \) in Cartesian coordinates
\[ A_x(\omega_x, \omega_y) = A(\omega, \phi), \quad (25) \]
where
\[ \omega_x = \omega \cos \phi, \quad (26a) \]
\[ \omega_y = \omega \sin \phi. \quad (26b) \]
6. Perform a 2D inverse Fourier transform of \( A_x(\omega_x, \omega_y) \) to obtain the image \( a(x, y) \),
\[ a(x, y) = \int_{\omega_y=-\infty}^{\infty} \int_{\omega_y=-\infty}^{\infty} A_x(\omega_x, \omega_y) e^{i2\pi(\omega_x x + \omega_y y)} d\omega_x d\omega_y. \quad (27) \]

To summarize, we use Eq. (18) to obtain equivalent parallel-beam projection from the fan-beam projection in the frequency domain, then use Metz and Pan’s parallel-beam algorithm to reconstruct the image.

### III. CONE-BEAM ALGORITHM

For the cone-beam geometry, the cone-beam focal point locations are distributed on the entire sphere of radius \( R \), covering a \( 4\pi \) solid angle. Projection data are obtained at every focal point location.

The reconstruction volume can be decomposed into many slices \( V_d(r, \phi, z) \) along an arbitrary direction \( \theta \) (see Fig. 3), where \( (r, \phi, z) \) are three circular cylindrical coordinates of the rotating circular cylindrical coordinate system and \( z \) is along the arbitrary direction \( \theta \). The corresponding activity distribution image in a given reconstruction volume is expressed as \( a_d(r, \phi, z) \).

We obtain image \( a_d(r, \phi, z) \)
by reconstructing image \( a_g(r_g, \phi_g, z_g) \) at different \( z_g \), that is, we reconstruct three-dimensional (3D) image slice-by-slice

\[
a_g(r_g, \phi_g, z_g) = a_g(r_g, \phi_g, z_g).
\]  

(28)

Reconstructing image \( a_g(r_g, \phi_g, z_g) \) in every slice is a 2D image reconstruction problem and we can use the fan-beam reconstruction algorithm to obtain image \( a_g(r_g, \phi_g, z_g) \). The fan-beam projection data of every slice \( z_g \) are obtained from the \( 4\pi \) acquisition cone-beam projection data. Then we can apply the new fan-beam reconstruction algorithm for every fixed \( z_g \) to obtain a 2D image \( a_g(r_g, \phi_g, z_g) \). Image \( a_g(r_g, \phi_g, z_g) \) is obtained by using Eq. (28). Expressing \( a_g(r_g, \phi_g, z_g) \) in Cartesian coordinates yields \( a_g(x_g, y_g, z_g) \). Note that the 3D image \( a_g(x_g, y_g, z_g) \) obtained in every direction \( \theta \) is an exact reconstruction of the activity \( a(x) \). For noiseless data, one direction \( \theta \) gives a complete reconstruction of the entire 3D reconstruction of the object. For noisy data, the final image \( a(x) \) is obtained by summing up all \( a_g(x_g, y_g, z_g) \) in all directions:

\[
a(x) = \int_{4\pi} a_g(x_g, y_g, z_g) d\theta.
\]  

(29)

This cone-beam algorithm is described in detail in the following steps.

1. Scale each attenuated cone-beam line-integral \( q(S, \alpha, \phi) \) by a factor \( e^{\mu D(S, \alpha, \phi)} \) obtaining

\[
n(S, \alpha, \phi) = e^{\mu D(S, \alpha, \phi)} n(S, \alpha, \phi),
\]  

(30)

where \( \mu \) is the linear attenuation coefficient and \( D(S, \alpha, \phi) \) is the distance defined in Fig. 2. Here \( S \) is a position function of a focal point which is on a sphere of radius \( R \), and \( S \) can be expressed as \( S(R, \theta_S, \phi_S) \). Angles \( \theta_S \) and \( \phi_S \) are the polar and azimuth angles of the focal point, respectively. The polar angle \( \alpha \) and the azimuth angle \( \phi \) point to the spatial direction of the projection ray through the focal point.

2. Obtain modified fan-beam projection data \( n(\sigma, \beta, z_g) \) of every slice \( z_g \) from the \( 4\pi \) acquisition modified cone-beam projection data \( n(S, \alpha, \phi) \). Angles \( \sigma \) and \( \beta \) are defined in Fig. 1, and \( \sigma \) is defined in Fig. 3. Equation \( n(\sigma, \beta, z_g) \) = \( n(S, \alpha, \phi) \) is satisfied with the following transformation of their variables:

\[
\phi = \beta,
\]

(31c)

\[
\alpha = \frac{\pi}{2} - \theta.
\]  

(31d)

3. Similar to Eq. (18), obtain \( M_k(\omega, z_g) \) by

\[
M_k(\omega, z_g) = \int_{-\pi}^{\pi} N_k(\sigma, z_g) e^{-i(2\pi R \sin(\sigma)+k\sigma)} d\sigma,
\]  

(32)

where

\[
N_k(\sigma, z_g) = \frac{1}{2\pi} \int_0^{2\pi} n(\sigma, \beta, z_g) e^{-ik\beta} d\beta,
\]  

(33)

and the \( z_g \) direction is along \( \theta \).

4. Shift and weight the frequency components to obtain the 2D Fourier transform of the 2D activity distribution function \( A_k(\omega, z_g) \),

\[
A_k(\omega, z_g) = \lambda \gamma(\omega_\mu)^4 M_k(\omega_\mu, z_g) + (1 - \lambda) \gamma(-\omega_\mu)^4 M_k(-\omega_\mu, z_g).
\]  

(34)

5. Perform an inverse Fourier transform of \( A_k(\omega, z_g) \),

\[
A(\omega, \phi, z_g) = \sum_{k=-\infty}^{\infty} A_k(\omega, z_g) e^{ik\phi}
\]  

(35)

in polar coordinates.

6. Express \( A(\omega, \phi, z_g) \) in Cartesian coordinates

\[
A(\omega, \phi, z_g) = A(\omega, \phi, z_g),
\]  

(36)

where

\[
\omega_x = \omega \cos \phi,
\]  

(37a)

\[
\omega_y = \omega \sin \phi.
\]  

(37b)

7. Perform 2D inverse Fourier transform of \( A(\omega, \phi, z_g) \) to obtain the image \( a_g(x) \).
TABLE I. Parameters of the modified Shepp-Logan phantom used in our numerical simulation. $\theta$, $\phi$, and $\gamma$ are three Euler angles.

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(8) Use Eq. (29) to obtain the reconstructed image $a(x)$.

IV. COMPUTER SIMULATIONS

In our computer-simulation studies, the projection data were generated from a modified 3D Shepp-Logan phantom with a uniform attenuator. The parameters of the mathematical phantom are shown in Table I. The cone-beam vertices were on a sphere that was centered at the origin and had a radius ($R=200$) units. There were 200 vertex locations on the sphere. The unit was in terms of the projection bin size. The uniform attenuator in the simulations had the same spherical properties than other methods in the family of Metz and Pan and Pan show that a quasioptimal method has better noise model.

The Metz-Pan solution in Eq. (20) yields a family of algorithms, depending on the parameter $\lambda$, which is defined as

$$
\lambda = \frac{[\gamma(\omega_n)]^{-nk}}{[\gamma(\omega_n)]^{-nk} + [\gamma(\omega_n)]^{2k}}.
$$

The Metz-Pan method is parametrized by an integer $n$, and Metz and Pan show that when $n=0$, for example, the Metz-Pan solution gives rise to the Tretiak-Metz algorithm. Metz and Pan also show that under certain assumptions on the noise model $n=2$ is a quasi-optimal choice because it minimizes the global variance in the reconstructed image. Metz and Pan show that a quasioptimal method has better noise properties than other methods in the family of Metz and Pan methods. In our computer simulations, the quasioptimal Metz-Pan algorithm was used, so that, setting $n=2$, Eq. (22) became

$$
A_k(x, y, z) = \int_{\omega_x=-\infty}^{\infty} \int_{\omega_y=-\infty}^{\infty} A_k(\omega_x, \omega_y, z) e^{2\pi i (\omega_x x + \omega_y y)} \times \omega_x d\omega_x d\omega_y.
$$

A. Fan-beam case

First, we modified each attenuated fan-beam line-integral $q(\sigma, \beta)$ by a scaling factor $e^{\mu D(\sigma, \beta)}$, obtaining $n(\sigma, \beta)$. For a given sampling frequency $\omega$, we calculated the shifted frequency $\omega_n = \sqrt{\omega^2 + (\mu/2 \pi)^2}$, then evaluated $M_k(\omega_n)$ using Eq. (18). We weighted $M_k(\omega_n)$ by using Eq. (40) to obtain $A_k(\omega)$. Finally, the image $a(x, y)$ was obtained by performing the inverse Fourier transform of $A_k(\omega)$.

B. Cone-beam case

First, we modified each attenuated cone-beam line-integral $q(\sigma, \beta, z)$ by a scaling factor $e^{\mu D(\sigma, \beta, z)}$, obtaining $n(\sigma, \beta, z)$. For a given sampling frequency $\omega$, we calculated the shifted frequency $\omega_n = \sqrt{\omega^2 + (\mu/2 \pi)^2}$, then evaluated $M_k(\omega_n, z)$ using Eq. (32) for an arbitrary direction $\theta$. We weighted $M_k(\omega_n, z)$ by using Eq. (41) to obtain $A_k(\omega, z)$. The image $a(x, y, z)$ is obtained by performing the inverse Fourier transform of $A_k(\omega_n, z)$. Finally, $a(x, y, z)$ was obtained by summing up the 200 $a(x, y, z)$ reconstructions. There were 129 fan-beam reconstructed slices in one direction, therefore $129 \times 200$ fan-beam reconstruction operations were required to reconstruct a 3D image from cone-beam projection data in 200 directions.
V. RESULTS

A. Fan-beam results

Figure 4 shows the reconstructed image from the fan-beam projection data. Figure 4(a) shows the fan-beam image reconstructed with attenuation compensation. Figure 4(b) shows the fan-beam image reconstructed without attenuation compensation by setting $\mu=0$ in the reconstruction algorithm, and Fig. 4(c) shows the central-slice of the true phantom.

Figure 5 shows the profiles of the reconstructed image and the true phantom in Fig. 4. The solid line indicates the true phantom, the dashed line shows the profile of the image reconstructed with attenuation compensation, and the dot-dashed line shows the profile of the image reconstructed without attenuation compensation. We can see that the profile with attenuation compensation closely matches the profile of the true phantom, while the profile without attenuation compensation deviates significantly from the truth as shown.

B. Cone-beam results

1. 3D reconstruction in one direction

Figures 6 and 7 show the 3D reconstruction images obtained in one direction $\theta$.

Figure 6 shows two cross-sectional images of the 3D reconstructions. Upper and lower rows are two vertical central sections of the 3D images. The quality of the reconstruction with attenuation compensation is good, and we can see the features of the phantom. Typical attenuation artifacts are visible in the reconstruction without attenuation correction.

To make a quantitative comparison, in Fig. 7 we depict intensity profiles taken across vertical lines of the images in Fig. 6. Here we can see that discrepancies between profiles with attenuation compensation and profiles of the actual phantom are small, while a comparison of profiles without attenuation correction and profiles of the true phantom show typical attenuation discrepancies.

2. 3D reconstruction with 200 directions

Figures 8 and 9 show the results of image reconstruction with 200 directions $\theta$.

Figure 8 shows three cross-sectional images of 3D reconstructions. The first column is the cone-beam reconstruction image in one direction with attenuation compensation, the second column is the combined cone-beam reconstruction image in 200 directions with attenuation compensation, and the last column is the true phantom.

Figure 9 shows a series of profiles of images in Fig. 8. Relatively larger errors are observed from the profiles obtained from the reconstructed image in one direction with attenuation compensation compared with the true phantom, and errors from the profiles obtained from the reconstructed image in 200 directions with attenuation compensation compared with the true phantom are smaller. The discrepancies between the images reconstructed in one direction and in 200 directions are small, and the images from these two methods are almost indistinguishable.
Fig. 7. (a)–(d) The four profiles of Fig. 6 drawn along the four white dashed lines from left to right in Fig. 6—(a)–(d) correspond to lines (1)–(4) of Fig. 6, respectively. Solid line is for phantom, the dashed line is for the reconstructed image with attenuator compensation, the dot-dashed line is for the reconstructed image without attenuation compensation.

3. 3D Reconstruction with the Noise Projection Data

Figures 10 and 11 show the 3D reconstruction images when the projections contain Poisson noise.

Figure 10 shows three cross-sectional images of 3D reconstructions. The first column is the cone-beam reconstruction image in one direction without attenuation compensation, the second column is the cone-beam reconstruction image in one direction with attenuation compensation, and the last column is the cone-beam reconstruction image in 200 directions, with attenuation compensation.

VI. Conclusion

Based upon a quasioptimal algorithm for the parallel-beam exponential Radon transform developed by Metz and Pan, we developed an exact analytical fan-beam image reconstruction algorithm that compensates for constant attenuation. The fan-beam method is based on Eq. (18), which builds the relationship between the parallel-beam exponential Radon transform and an exponential fan-beam Radon transform in the frequency domain, utilizing properties of the delta function.

Figure 11 shows a series of profiles of images in Fig. 10. As shown in Figs. 10 and 11, the images reconstructed in one direction are much noisier than those reconstructed using 200 directions.
A cone-beam reconstruction method was also developed by concatenation of the fan-beam reconstructions and by direction-by-direction summation. This cone-beam reconstruction method only applies to $4\pi$ solid angle data acquisition geometry. It is not extendable to more realistic imaging geometries. Our fan-beam and cone-beam algorithms have been implemented and the computer simulations are provided.

For noiseless cone-beam imaging we can obtain an exact reconstruction by using an arbitrary direction $\theta$. The use of many directions and summing the reconstructions obtained at each direction $\theta$ is a better choice when noise is present in projection data. This is verified in Figs. 10 and 11. Noise properties of the proposed algorithms are only addressed briefly in this paper, extending the noise investigation of Metz and Pan’s algorithms to fan-beam and cone-beam cases will be conducted in our future research.

The challenge for the future is to develop a filtered back-projection algorithm with a spatially invariant filter for cone-beam projections with attenuation compensation. The $4\pi$ solid angle imaging geometry is not realistic. Future research will be focused on developing efficient cone-beam algorithms with attenuation correction using realistic focal point orbits, for example, helical cone-beam focal point orbits.

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FIG. 11. Profiles of the images along three directions of Fig. 10: The solid line is the profile of column (c) of Fig. 10, the dashed line shows the profile of column (b) of Fig. 10, and the dot-dashed line shows the profile of column (a) of Fig. 10: (a) is drawn along the dashed line 1 in Fig. 10, (b) is drawn along the dashed line 2 in Fig. 10, (c) is drawn along the dashed line 3 in Fig. 10.

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