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An FDK-like cone-beam SPECT reconstruction algorithm for non-uniform attenuated projections acquired using a circular trajectory

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Abstract
In this paper, Novikov’s inversion formula of the attenuated two-dimensional (2D) Radon transform is applied to the reconstruction of attenuated fan-beam projections acquired with equal detector spacing and of attenuated cone-beam projections acquired with a flat planar detector and circular trajectory. The derivation of the fan-beam algorithm is obtained by transformation from parallel-beam coordinates to fan-beam coordinates. The cone-beam reconstruction algorithm is an extension of the fan-beam reconstruction algorithm using Feldkamp–Davis–Kress’s (FDK) method. Computer simulations indicate that the algorithm is efficient and is accurate in reconstructing slices close to the central slice of the cone-beam orbit plane. When the attenuation map is set to zero the implementation is equivalent to the FDK method. Reconstructed images are also shown for noise corrupted projections.

1. Introduction

Single photon emission computed tomography (SPECT) enables us to visualize functional information on human organs or body systems with the use of radioactive isotopes (radiopharmaceuticals) introduced into the body. Unlike x-ray CT, SPECT provides information on organ function, not just structure, and thus can be used to diagnose diseases and cancers much earlier. SPECT can be used in imaging of the bone, heart, liver/spleen,
brain, renal, thyroid and other organs. This work concerns three-dimensional (3D) image reconstruction in SPECT using cone-beam collimators with a circular trajectory.

Cone-beam CT with a circular scanning trajectory is known for its fast data acquisition and minimum mechanical complexity. Much work has been done about 3D cone-beam reconstruction in the past two decades (Feldkamp et al. 1984, Grangeat 1991, Smith 1985, Tuy 1983), among which the FDK method (Feldkamp et al. 1984) is famous for its simplicity of implementation. In SPECT, however, the attenuation should be compensated for in quantitative studies. The most representative works in image reconstruction in 3D SPECT include different inversion algorithms for 3D exponential parallel-beam transform (Palamodov 1996, Wagner and Noo 2001, Kunyansky 2004). Zeng et al. (1991) developed an algorithm for 3D cone-beam image reconstruction with non-uniform attenuation map; however the algorithm was an iterative one. To date, the analytical 3D cone-beam reconstruction with non-uniform attenuation has not yet been established. Thanks to Novikov’s explicit inversion formula (Novikov 2002, Natterer 2001) for attenuated 2D Radon transform, the 3D reconstruction with non-uniform attenuation becomes feasible. By a change of variables, Novikov’s formula can be applied to fan-beam geometry. Cone-beam reconstruction is then straightforward by extending the 2D implementation to a 3D implementation. When attenuation is zero, our cone-beam reconstruction is equivalent to the FDK method.

It is well known, unfortunately, that cone-beam projections acquired with a circular trajectory are not sufficient for exact reconstruction according to Tuy’s data sufficiency condition (Tuy 1983). Therefore, in this paper the reconstruction method is an approximation. Our cone-beam reconstruction is efficient and is accurate in the regions close to the cone-beam trajectory plane.

This paper is organized as follows. In section 2, Novikov’s inversion formula for the attenuated 2D Radon transform is reviewed and then applied to the fan-beam geometry by changing variables. An FDK-like cone-beam SPECT reconstruction algorithm is then derived based on the fan-beam SPECT reconstruction algorithm. Section 3 provides simulation results to validate these algorithms. Conclusions are given with a short discussion in section 4.

2. Algorithm

2.1. Novikov’s inversion formula

Assume \( \mathbf{x} = (x, y) \) is a vector in a two-dimensional Euclidean space. Let \( f(\mathbf{x}) \) denote the distribution of radiopharmaceutical concentration and \( \mu(\mathbf{x}) \) the attenuation map of the tissue, which can be non-uniform. The mathematical model of the attenuated 2D Radon transform in a parallel-beam geometry (see figure 1) is (Gullberg 1979)

\[
(R_{\mu} f)(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\hat{\theta}^\perp) e^{-\left(D_{\mu}(s\theta + t\hat{\theta}^\perp, -\theta \perp)\right)} \, dt,
\]

where \( \theta = (\cos \theta, \sin \theta) \), \( \hat{\theta}^\perp = \left(-\sin \theta, \cos \theta\right) \) and \( (D_{\mu}(\mathbf{x}, \phi)) \) is the divergent beam transform of \( \mu(\mathbf{x}) \) in the direction of \( \phi = (\cos(\phi), \sin(\phi)) \) defined as \( (D_{\mu}(\mathbf{x}, \phi)) = \int_{0}^{\infty} \mu(\mathbf{x} + p\phi) \, dp \). Without attenuation, equation (1) becomes the 2D Radon transform \( (R f)(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\hat{\theta}^\perp) \, dt \).

Assuming \( \mu(\mathbf{x}) \) is known, Novikov (2002) gave an explicit inversion formula to reconstruct \( f(\mathbf{x}) \) from the parallel projection data \( g(\theta, s) = (R_{\mu} f)(\theta, s) \):

\[
f(\mathbf{x}) = \frac{1}{4\pi} \text{Re} \left\{ \nabla \cdot \int_{0}^{2\pi} \left[ e^{-h(\theta, s)}(D_{\mu}(\mathbf{x}, \phi))^{-1} e^{-h(\theta, s)} H e^{h(\theta, s)} g(\theta, s) \right] \, d\theta \right\}.
\]
where
\[ h(\theta, s) = \frac{1}{2}(I + iH)(R\mu)(\theta, s), \quad H\psi(\theta, s) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\psi(\theta, s')}{s - s'} \, ds', \]
with \( i^2 = -1 \). \( I \) is the identity operator. \( H \) is the Hilbert transform with respect to the second parameter throughout this paper and \( \text{p.v.} \) denotes the Cauchy principal value of the integral. The symbol \( \nabla \) stands for the divergence defined as
\[ \nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \]
for vector \( V = (V_x, V_y) \).

2.2. Fan-beam reconstruction with equal detector spacing

Fan-beam data acquisition geometry with equally spaced detectors is shown in figure 2.

The fan-beam focal point \( S \) is located on a circle of radius \( D \). Without loss of generality, let us put the detector passing through the centre \( O \) and perpendicular to \( SO \). Denote the projection data acquired at position \( u \) on the detector with view angle \( \beta \) by \( \tilde{g}(\beta, u) \), where the operator \( (\tilde{D}_\mu f)(\beta, u) \) is given by
\[ (\tilde{D}_\mu f)(\beta, u) = \int_0^\infty f(x_S + t\alpha_\mu) e^{-(D\mu)(x_S + t\alpha_\mu, \alpha_\mu)} \, dt. \]
\( x_S = (x_S, y_S) \) is the focal point location and \( \alpha_\mu \) is the unit vector shown in figure 2. It is clear that \( \tilde{g}(\beta, u) \) is equal to projection data \( g(\theta, s) \) acquired at \( (\theta, s) \) in the parallel-beam geometry if
\[ s = \frac{uD}{\sqrt{u^2 + D^2}}, \quad \theta = \beta + \tan^{-1} \left( \frac{u}{D} \right), \]
i.e., we have
\[ g \left( \beta + \tan^{-1} \left( \frac{u}{D} \right), \frac{uD}{\sqrt{u^2 + D^2}} \right) = \tilde{g}(\beta, u). \]
You et al proved that the Hilbert transform of any non-attenuated parallel-beam data $k(\theta, s)$ with respect to the second variable $s$ can be evaluated by the following integral of their fan-beam counterpart $\tilde{k}(\beta, u)$ (You et al 2005)

$$Hk(\theta, s) = \frac{1}{\pi} \text{p.v.} \int_{-\pi/2}^{\pi/2} \frac{\tilde{k}(\beta, D \tan(\sigma'))}{\sin(\sigma - \sigma')} \, d\sigma',$$

where (5) is assumed and $\sigma = \tan^{-1}\left(\frac{u}{D}\right)$. Changing variable $\sigma'$ to $u'$, with $u' = D \tan(\sigma')$ and $du' = \frac{u'^2 + D^2}{D} \, d\sigma'$, (7) becomes

$$Hk(\theta, s) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\tilde{k}(\beta, u')}{\sqrt{u'^2 + D^2}} \frac{D}{u'^2 + D^2} \, du' = \frac{\sqrt{u'^2 + D^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{u'^2 + D^2}} \, du' = \tilde{H}\tilde{k}(\beta, u),$$

where $\tilde{H}$ is the Hilbert transform in the fan-beam geometry and is defined as

$$\tilde{H}\tilde{\phi}(\beta, u) = \sqrt{u'^2 + D^2} H \left( \tilde{\phi}(\beta, u) \frac{1}{\sqrt{u'^2 + D^2}} \right).$$

Changing variable in (2), with $d \theta = D^3(D^2 + u^2)^{-3/2} \, du \, d\beta$, we obtain the inversion formula for the fan-beam geometry as follows:

$$f(x) = \frac{1}{4\pi} \text{Re} \left\{ \nabla \cdot \left[ \int_0^{2\pi} \frac{2\pi}{U} \left\{ \tilde{\beta} e^{-\tilde{k}(\beta, u) + (D\beta)(\frac{u}{D})} H \left( \frac{\tilde{E}(\beta, u)}{D^2 + u^2} \right) \right\}_{u = u'} \, d\beta \right\} \right\}$$

with $U = \frac{D + x \sin \beta - y \cos \beta}{D}$,

$$\tilde{\beta} = \begin{pmatrix} \cos(\beta + \tan^{-1}\frac{u}{D}) \\ \sin(\beta + \tan^{-1}\frac{u}{D}) \end{pmatrix}.$$
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Figure 3. Cone-beam geometry.

\[ \tilde{h}(\beta, u) = \frac{1}{2} (I + i\tilde{H})(\tilde{D}\mu)(\beta, u) = \frac{1}{2} (I + i\tilde{H}) \int_{0}^{\infty} \mu(\bar{z}, t_{z}) \, dt, \]

\[ u' = \frac{D(x \cos \beta + y \sin \beta)}{D + x \sin \beta - y \cos \beta}. \]

The above inversion formula can be discretized to reconstruct the image from fan-beam projections. The Hilbert transform and divergence operator are implemented as two separate filtrations. Direct discretization of (10) results in a filtering–backprojection-filtering algorithm.

2.3. Cone-beam reconstruction with circular scanning trajectory

The cone-beam geometry is described in figure 3. The cone-beam focal point \( S \) is on a circle centred at \( O \) with radius \( D \) and in the plane \( z = 0 \). Assume the detector plane is perpendicular to \( SO \) at \( O \). Define the coordinates system \( (u, \xi) \) centred at \( O \) on the detector plane such that the \( u \)-axis is parallel to the tangent of the trajectory and the \( \xi \)-axis is parallel to the \( z \)-axis (i.e. the axis of rotation). Denote \( \tilde{g}(\beta, u, \xi) \) as the attenuated projection data acquired at view angle \( \beta \) and at point \( (u, \xi) \) on the detector plane and \( (\tilde{D}\mu)(\beta, u, \xi) \) as the unattenuated projection of the attenuation map.

Introducing the rotated coordinates

\[
\begin{align*}
\begin{cases}
 s = x \cos \beta + y \sin \beta \\
 t = -x \sin \beta + y \cos \beta \\
 z = z
\end{cases}
\end{align*}
\]

and following the derivation of the FDK method (Feldkamp et al 1984), we obtain the reconstruction formula at \( x = (x, y, z) \) as

\[
f(x) = \frac{1}{4\pi} \operatorname{Re} \left\{ \nabla \cdot \int_{0}^{2\pi} \frac{D^2 \sqrt{D^2 + \xi^2}}{D - t} \frac{e^{-\tilde{h}(\beta, u, \xi)(\tilde{D}\mu)(\beta, u, \xi)}}{\tilde{H}} \frac{e^{\tilde{h}(\beta, u, \xi)}}{D^2 + u^2 + \xi^2} \bigg|_{u = u'} \, d\beta \right\}
\]

(12)
with
\begin{align}
\hat{\beta} &= \left( \cos (\beta + \tan^{-1} \frac{u}{\sqrt{D^2 + \xi^2}}), \\
&\quad \sin (\beta + \tan^{-1} \frac{u}{\sqrt{D^2 + \xi^2}}) \right), \quad (13)
\end{align}
\begin{align}
\hat{h}(\beta, u, \xi) &= \frac{1}{2} (I + i \hat{\nu} (D \mu)(\beta, u, \xi)), \quad (14)
\end{align}
\begin{align}
u'' &= \frac{Ds}{D - t}, \quad (15)
\end{align}
\begin{align}\xi &= \frac{Dz}{D - t}, \quad (16)
\end{align}
\begin{align}
\hat{H} k(\beta, u, \xi) &= \sqrt{u^2 + D^2 + \xi^2} H \frac{k(\beta, u, \xi)}{\sqrt{u^2 + D^2 + \xi^2}}. \quad (17)
\end{align}

This formula gives an approximate algorithm to reconstruct an image from cone-beam projection data. Artefacts are introduced in the reconstructed image except for the central slice. However, the inversion formula is easy to implement and efficient.

The reconstruction of \( f(\mathbf{x}) \) from formula (12) can be implemented in the following three steps:

- **Step 1. Filtration**
  (i) Multiply the projection data, \( \hat{g}(\beta, u, \xi) \), by the \( e^{i \hat{\beta}(\beta, u, \xi)} \) to obtain the modified projection data. Weight the modified projection data by a factor \( \frac{1}{\sqrt{D^2 + u^2 + \xi^2}} \).
  (ii) Filter the weighted modified projection data by doing the Hilbert transform defined in section 2.1.
  (iii) Post-weight the result from (ii) by multiplying by \( \hat{H} e^{-i \hat{\beta}(\beta, u, \xi) + (D \mu)(\beta, u, \xi)} \) to obtain a vector \( \hat{Q} \).

- **Step 2. Backprojection**
  Weight \( \hat{Q} \) by \( \frac{D^2 \sqrt{D^2 + u^2 + \xi^2}}{D - t} \) and backproject it over the 3D reconstruction grid.
Step 3. Filtration

The divergence can be seen as a filtration after the backprojection. Keep the real component of the result and scale it by $\frac{1}{4\pi}$.

The Hilbert transform and divergence operator are both one-dimensional (1D) filtration performed in the spatial domain. The backprojection is performed as a 3D cone-beam backprojection. The implementation is thus a filtering–backprojection-filtering algorithm. In the case where there is no attenuation, $\widetilde{h}(\beta, u, \xi)$ and $(D\mu)(\vec{x} - \vec{\beta})$ both vanish. Exchanging the order of divergence and backprojection (Natterer 2001) and combining the derivative and the Hilbert transform together, we obtain a 1D ramp filtration along the row direction in the detector plane. Our algorithm is thus reduced to the FDK method if $\mu(\vec{x}) = 0$. 

Figure 5. Emission phantom for the fan-beam study.

Figure 6. Attenuated fan-beam projections.
3. Numerical results

In our computer simulation, for the fan-beam geometry, we choose the attenuation map as in figure 4 and the emission phantom as in figure 5. The focal length $D$ is set to be 128 units where one unit is the image voxel size. The attenuated noise-free projections with 128 views over $360^\circ$ and the reconstructed image in a $128 \times 128$ array are shown in figures 6 and 7, respectively. This simulation result validates the reconstruction formula (10) derived for fan-beam projections.

For the cone-beam geometry, the attenuation map is defined in a $129 \times 129 \times 129$ array with maximum coefficient of 0.02 per unit and is composed of two spheres. The central slice of the attenuation map is shown in figure 8. If we choose one unit as 0.17 cm, the image is approximately the size of a human skull and the attenuation coefficient 0.02 per unit is the same as that of water at 140 keV. The noise-free projections are analytically generated at 128 views.
Figure 9. Original emission phantom (first row), reconstructed image from noise-free projections (second row) and reconstructed image from noisy projections (third row) for $z = 0$ (first column), $x = 0$ (second column) and $y = 0$ (third column).

Figure 10. Profiles: dashdot line is for the original phantom, solid line for reconstructed image and dashed line for reconstructed image without attenuation correction.

over 360° and at $129 \times 129$ equally spaced detector grid. The focal length $D$ is set to be 120 units. The reconstructed images at the three orthogonal central slices are shown in figure 9.

Due to the random nature of radioactivity and measurement error, there is inevitable noise in the acquired data. We model the noise in the projections with Poisson distributions. Figure 9 also shows the reconstructed images at the three orthogonal central slices for noisy
projections. The Poisson noise is such that the total count calculated from projections of the $x$–$y$ plane is $10^6$. Profiles drawn across the phantom and reconstructed image are shown in figure 10 for a noise-free case and figure 11 for a noisy case.

4. Conclusion

In this paper we derived an FDK-like cone-beam reconstruction algorithm for non-uniform attenuated projections acquired using a circular trajectory. This algorithm is a filtering–backprojection-filtering algorithm and gives an approximate reconstruction of the image. Simulation results showed the efficiency and accuracy of the reconstruction from noise-free projection data.

From Tuy’s data sufficiency condition, we know it is impossible to reconstruct exactly the image from projections acquired in the geometry described above. However, in regions close to the orbit plane images can be reconstructed with acceptable artefacts. In order to do exact reconstruction, a non-planar trajectory should be adopted. Adding either an arc or a vertical line to the circular trajectory is a solution. Using a helix trajectory is another. Future work is required to extend Novikov’s formula to other trajectories to obtain exact reconstruction.

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