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Diffusion tensor MR imaging of principal directions: a tensor tomography approach

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Abstract

A novel approach to reconstructing the principal directions of a diffusion tensor field directly from magnetic resonance imaging (MRI) data using a tensor tomography data acquisition approach was developed. If tensor eigenvalues are assumed to be known, the reconstruction of principal directions requires fewer measurements than the reconstruction of the full tensor field. The tensor tomography data acquisition method (rotating diffusion gradients) leads to a unique reconstruction of principal directions, whereas the conventional MRI acquisition technique (stationary diffusion gradients) leads to an ambiguous reconstruction of principal directions when the same number of measurements are used. A computer-generated phantom was used to simulate the diffusion tensor field in the mid-ventricular region of the myocardium. The principal directions of the diffusion tensor field were assumed to align with the fibre structure of the myocardium. An iterative algorithm was used to reconstruct the principal directions. Computer simulations verify that the proposed method provides accurate reconstruction of the principal directions of a diffusion tensor field.

1. Introduction

Magnetic resonance imaging (MRI) has been shown to be effective for imaging diffusion tensor fields through a process known as diffusion tensor MRI (DT MRI) (Basser et al 1994, Pierpaoli et al 1996). The goal behind DT MRI is to reconstruct diffusion tensor components prior to their eigenvalue decomposition. In some imaging applications, a complete understanding of the tensor field itself is of secondary interest. For example, in cardiac imaging (Edelman et al 1994, Reese et al 1995, 1996, Tseng et al 1999, 2000), the primary objective is to use DT MRI to determine the principal direction of the diffusion tensor field, which

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corresponds to the eigenvector with maximum eigenvalue. Recently a new method, diffusion tensor tomography MRI (DTT MRI) (Gullberg et al 1999, 2001), was proposed. With this method tensor fields are reconstructed from measurements of projections of the tensor fields. The focus of this paper is to show that DTT MRI is a more effective way of estimating the principal directions than conventional DT MRI, if the eigenvalues of the tissue are known.

MRI diffusion imaging has been demonstrated to have many important applications in diagnostic medicine. For example, diffusion imaging has been widely used to detect cerebrovascular ischemia. Several reviews (Van Bruggen et al 1994, Fisher et al 1995, Hossmann and Hoehan-Berlage 1995, Schabitz et al 1995) suggest that diffusion imaging offers the potential for early, noninvasive detection of stroke and reliable prediction of ischemic brain injury. It was shown early on in animals that the apparent diffusion coefficient (ADC) of tissue water decreased dramatically after acute cerebral ischemia (Moseley et al 1990). More recently, DT MRI has shown that the principal diffusivity (Scollan et al 1998) and the trace (Moseley et al 1995, Warach et al 1995) of the diffusion tensor can be important predictors of the onset of stroke.

In cardiac imaging it has been verified that myocardial ischemia results in a reduction of tissue diffusivity, although with a different time course than in the brain (Hsu et al 1998b). Also, it has been established that for isolated perfused myocardium the water diffusion anisotropy measured by MRI faithfully parallels histologic anisotropy. Knowledge of the principal directions of the tensor field provides myocardial fibre organization (Scollan et al 1998, Hsu et al 1998a). Myocardial fibre architecture is a key determinant of the electrical and mechanical properties of the myocardium (Tuch et al 1998, Chen et al 1993, Kanai and Salama 1995, Guccione et al 1991, Bovendeerd et al 1992, Hunter et al 1997). Some diseases can be characterized by abnormal fibre structure and thereby deduce abnormal electrical pathways (Chen et al 1993), even though the tissue properties may be normal.

Here we consider a cardiac imaging application. We will address the question of whether one can estimate principal directions directly from DTT MRI data, thereby possibly reducing the number of measurements associated with conventional DT MRI. As will be shown, the problem can be formulated in terms of the estimation of eigenvectors and eigenvalues instead of tensor components. Nevertheless, the estimation of tensor components seems to be an easier task than direct estimation of eigenvalues and eigenvectors. So far, we have not yet been able to separate the reconstruction of eigenvalues from the reconstruction of eigenvectors. The incorporation of a priori information in terms of known eigenvalues reduces the number of measurements needed to estimate principal directions directly from DTT MRI data.

It can be assumed that the eigenvalues of the diffusion tensor in cardiac tissue are known. It was found that these values are uniform across the myocardium, and they are similar to the diffusivity reported in other human tissues, which is less than half the diffusivity of water at 37 °C (Reese et al 1995). The secondary anisotropy, with all eigenvalues having different values, was observed. We acknowledge that the validity of assuming known eigenvalues will require further validation by other researchers in the DT MRI field.

This paper concentrates on two tasks. The first is presentation of a novel approach to reconstructing principal directions directly from DT MRI data, assuming known eigenvalues. Because a priori information is incorporated, the reconstruction of principal directions requires fewer measurements than in the case of reconstructing a full tensor field. This may be an important asset, since some DT MRI measurements are strongly affected by systematic errors such as eddy current artifacts (Seifert et al 2000). Reducing the number of measurements may be beneficial when only artifact-free measurements are sought. The second goal of this paper is to show that the diffusion tensor tomography MRI (DTT MRI) (Gullberg et al 1999, 2001) approach is a more effective and flexible means of reconstructing principal directions.
Figure 1. The projection geometry. The slice selective acquisition was used, which means that the projection line is always in the XY plane. The 3D vector \( \vec{\omega} \) was parametrized by two angles: \( \phi \) and \( \theta \).

than standard DT MRI techniques. In this paper projection reconstruction (PR) imaging (Glover and Pauley 1992) will be used to evaluate standard DT MRI techniques, thus, at times we will use the acronym PR DT MRI.

2. Theory

2.1. Diffusion tensor MRI imaging: tensor tomography versus conventional MRI

In PR imaging, radial lines are acquired in Fourier space instead of rectilinear lines, as is the case in 2D Fourier transform imaging methods such as echo-planar imaging (EPI). The PR signal during readout in DT MRI can be expressed as (Gmitro and Alexander 1993)

\[
s_{\vec{\omega}}(t) = \int \rho(\vec{x}) e^{i\gamma \vec{G}_r \cdot \vec{\omega}_1} e^{-\gamma \vec{G}_r (\vec{\omega}_2^T D \vec{\omega}_2) \Lambda} e^{-b(\vec{\omega}_1 \cdot \vec{x})} d\vec{x}
\]

where \( \vec{G}_r \) is the readout gradient, \( \rho \) is the spin density, \( \gamma \) is the gyromagnetic ratio, \( G \) is the amplitude of the diffusion-weighting gradient, \( \vec{\omega} \) is the direction of the applied diffusion-weighting gradient, \( \Lambda \) is the length of one lobe of the diffusion pulse, \( D \) is the diffusion tensor, and \( \Delta \) is the separation between the starting point of each of the two gradient pulses.

Taking the Fourier transform of \( s_{\vec{\omega}}(t) \) in equation (1) and applying the well-known Fourier section theorem, one can show that the measurements yield the following quantity:

\[
p_{\vec{\omega}}(\theta, \xi) = \int \rho(\vec{x}) e^{-b(\omega^T D \vec{x} \omega)} \delta(\vec{x} \cdot \vec{\theta} - \xi) d\vec{x},
\]

where \( b \) is a positive constant, \( \vec{\theta} \) is the direction of the readout gradient, and \( \xi \) is the coordinate of the sampled projection at the angle \( \vec{\theta} \) (Gullberg et al 1999). According to equation (2), the PR signal can be presented as a ‘projection’ \( p_{\vec{\omega}}(\theta, \xi) \), (which is proportional to the Fourier transform of \( s_{\vec{\omega}}(t) \) in equation (1)) of the two functions \( \rho \) and \( D \).

In the following analysis we will restrict our attention to slice-by-slice data acquisition, in which the readout direction is defined by the projection angle \( \theta \). The necessary information is provided by datasets with different choices of \( \vec{\omega} \). The 3D direction of this vector is shown in figure 1.
The goal behind diffusion tensor MRI is the reconstruction of $D$. In conventional DT MRI measurements are made with stationary diffusion gradients, i.e., $\vec{\omega}$ is a constant vector. The same diffusion-weighted spin density function is either projected at every angle $\theta$ given by equation (2), as in the case of PR, or acquired line by line in Fourier space, as in the case of EPI. The standard reconstruction technique for PR DT MRI is the filtered backprojection (FBP) method. For EPI the standard reconstruction technique is the Fourier inversion formula. Exponential terms of $D$ weighted by the spin density $\rho$ are reconstructed and $D$ is obtained by taking the logarithm of the ratio of the diffusion-weighted image and the spin density image. The spin density image is estimated using the same imaging sequence, but with the diffusion gradient equal to zero. Conventional DT MRI therefore provides reconstruction of $\vec{\omega}^T D \vec{\omega}$ for a fixed $\vec{\omega}$. To solve for the 3D diffusion tensor $D$ it is necessary to obtain six independent datasets by obtaining measurements of six non-collinear diffusion-weighting directions $\vec{\omega}$. Through appropriate choices of $\vec{\omega}$, conventional DT MRI reconstructs the diffusion tensor components $D_{xx}, D_{yy}, D_{zz}, D_{xy}, D_{xz}$ and $D_{yz}$ from the six datasets.

The diffusion tensor tomography MRI (DTT MRI) approach is similar to the PR DT MRI technique. However, instead of using fixed diffusion gradients $\vec{\omega}$ in equation (2) DTT MRI uses rotating gradients where $\vec{\omega}$ is a function of $\theta$. In order to reconstruct a 3D tensor field, the six datasets for the six different $\vec{\omega}$s are used simultaneously during the reconstruction process. Because equation (2) is an exponential projection of the diffusion tensor distribution $D$, a special iterative reconstruction technique is required (Panin et al. 2000).

2.2. 2D case

The main idea of reconstructing the distribution of the principal eigenvectors can easily be demonstrated in a 2D case. The 2D tensor for each pixel can be represented in terms of known eigenvalues $\lambda_1$ and $\lambda_2$, and unknown eigenvectors $\vec{V}_1 = (V_{1,1}, V_{1,2})$ and $\vec{V}_2 = (V_{2,1}, V_{2,2})$:

$$D_{ij} = \lambda_1 V_{i,j} V_{1,j} + \lambda_2 V_{i,j} V_{2,j} \quad \lambda_1 > \lambda_2 > 0 \quad i, j = 1, 2.$$ (3)

The unknown orthogonal unit eigenvectors can be parametrized by the angle $\Phi$:

$$\vec{V}_1 = [\cos \Phi, \sin \Phi]^T \quad \vec{V}_2 = [-\sin \Phi, \cos \Phi]^T \quad 0 \leq \Phi < \pi.$$ (4)

Since the eigenvectors are defined up to a sign, it can be seen that $\Phi$ and $\Phi + \pi$ are equivalent. Since we assume that $\lambda_1$ and $\lambda_2$ are known, there is only one unknown variable function $\Phi(\vec{x})$. We can expect that only a single set of measurements of diffusion-weighted gradients is sufficient to reconstruct the principal directions.

As mentioned above, conventional DT MRI provides a reconstruction of $\vec{\omega}^T D \vec{\omega}$ for each pixel $\vec{x}$, where the diffusion gradient direction can be parametrized by a fixed angle $\phi$: $\vec{\omega} = [\cos \phi, \sin \phi]^T$. (Note, when considering DT MRI, the diffusion gradient direction is fixed and is not a function of the readout direction. Later we discuss applications of DTT MRI in which the diffusion gradient direction is a function of the readout direction $\theta$, which we parametrize as $\phi(\theta)$.) The expression $\vec{\omega}^T D \vec{\omega}$ can be written in terms of $\phi$:

$$\vec{\omega}^T D \vec{\omega} = \lambda_1 \cos^2(\phi - \phi) + \lambda_2 \sin^2(\phi - \phi).$$ (5)

When the data are consistent, the reconstructed quantity $\vec{\omega}^T D \vec{\omega}$ must satisfy for each $\vec{x}, \lambda_1 \geq \vec{\omega}^T D \vec{\omega} \geq \lambda_2$. When this condition holds, equation (5) admits two solutions for $\Phi$:

$$\Phi_1 = \phi + \arctan \left( \frac{\lambda_1 - \vec{\omega}^T D \vec{\omega}}{\vec{\omega}^T D \vec{\omega} - \lambda_2} \right) + \pi k_1 \quad \text{if} \quad \vec{\omega}^T D \vec{\omega} - \lambda_2 \neq 0$$ (6)
Diffusion tensor MR imaging of principal directions: a tensor tomography approach 2741

$$\Phi_2 = \phi + \arctan\left( -\frac{\lambda_1 - \bar{\omega}^T \bar{D} \bar{\omega}}{\bar{\omega}^T \bar{D} \bar{\omega} - \lambda_2} \right) + \pi k_2 \quad \text{if} \quad \bar{\omega}^T \bar{D} \bar{\omega} - \lambda_2 \neq 0 \quad (7)$$

and

$$\Phi_{1,2} = \phi + \frac{\pi}{2} + \pi k_3 \quad \text{if} \quad \bar{\omega}^T \bar{D} \bar{\omega} - \lambda_2 = 0 \quad (8)$$

where integers $k_1$, $k_2$ and $k_3$ are chosen to ensure that $0 \leq \Phi_1, \Phi_2 < \pi$. A single measurement of diffusion-weighted gradients generally provides two different $\Phi$ at each space point. Therefore, the principal directions are not uniquely defined and there are a variety of tensor fields that satisfy the same measurement.

We present here an empirical conjecture that the DTT MRI method can overcome this problem. In the DTT MRI method we do not reconstruct $\bar{\omega}^T \bar{D} \bar{\omega}$, because $\phi$ is not fixed. Instead, we directly reconstruct the entire tensor field $D$. For each point in space we find $\Phi$ that fits the projection measurements for all the angles $\theta$ that correspond to the rotated readout gradient along $\theta$ and the diffusion gradient along $\bar{\omega}(\phi(\theta))$. One can express equations (6) and (7) as $\Phi_{1,2} = \phi + \arctan(|\tan(\Phi - \phi)| + k\pi)$ where $\Phi$ is the true angle of the eigenvector at some point in space. One solution is $\Phi + k\pi$ and a second solution is $2\phi - \Phi + k\pi$. Therefore, in a least squares sense there should not be duality of $\Phi$ because there is only one solution that satisfies equation (5) for all $\phi$; the other solution only satisfies equation (5) for a few $\phi$. Therefore, in the DTT MRI method only one set of projection measurements over $\theta$, corresponding to one set of $\bar{\omega}(\phi(\theta))$ measurements, should be sufficient to define $\Phi$ for any given point. Whereas from our above arguments multiple $\bar{\omega}(\phi)$ measurements (i.e. for different $\phi$) are needed for the conventional DT MRI method.

Careful observation reveals that in using the DTT MRI approach there may be multiple tensor fields that give identical projections. We will present examples of totally different tensor fields that produce the same projection function $p_{\omega}(\theta, \xi)$ in equation (2). The derivation is presented in appendix A and figure 2 gives examples of ambiguous tensor fields for $\phi = \theta$ (hereafter labelled or referred to as ‘plus’ acquisition), and for $\phi = -\theta$ (referred to as ‘minus’ acquisition). In the ‘minus’ acquisition the projection readout gradient rotates, as in the ‘plus’ acquisition, but the diffusion gradient rotates at an equal angle in the opposite direction. There are significant differences in comparison to conventional DT MRI (stationary $\bar{\omega}$), where the ambiguity is point by point in space. On the other hand, in the DTT MRI case the ambiguity is functional. For example, the equivalent contribution to the projection comes from different space points of ambiguous tensor fields, as illustrated in figure 2.

Another difference between DT MRI and DTT MRI is that the change of acquisition changes the ambiguity situation for DTT MRI: tensor fields that are ambiguous for ‘plus’ acquisitions are well defined for ‘minus’ acquisitions. This is the case for the examples in figure 2. The phantoms in figure 2((a) left and right) produce the same projections for the ‘plus’ acquisition, whereas, the ‘minus’ acquisition produces non-ambiguous projections. Likewise, the phantoms in figure 2((b) left and right) produce the same projections for the ‘minus’ acquisition, whereas, the ‘plus’ acquisition produces a non-ambiguous solution. This demonstrates the flexibility of DTT MRI. In comparison (for the 2D phantoms considered) there is no beneficial acquisition with conventional DT MRI: the phantoms all produce underdetermined measurements due to the symmetry. Note that we considered only the linear relationship between $\phi$ and $\theta$. There are a variety of acquisition schemes that could be used. Practically, it is possible to choose one acquisition scheme, which should work fine for the expected diffusion fields.
Figure 2. Two examples of tensor fields that provide equivalent projections in the DTT MRI method for (a) the two tensor fields with uniform spin density that give the same projections for the ‘plus’ acquisition \( \phi = \theta \), and (b) the two tensor fields with uniform spin density that give the same projections for the ‘minus’ acquisition \( \phi = -\theta \). The contributions equivalent (equivalent \( |\vec{\omega} V_{1}| \)) to the projections are marked by points. These points are interchanged in space for these two examples. Other tensor fields can be constructed by modifying the ‘rings’ (fixed \( r \)) taken from these two examples.

One can argue, based on the derivation in appendix A and examples in figure 2, that it is unlikely in practice that such symmetrical fields will exist. Therefore, we can expect that the estimation of the principal direction will not be ambiguous, except for some rare cases. Our results in section 4 show that perturbations from ambiguous tensor fields might remove ambiguities. Also, it is highly likely that any perturbation from a non-ambiguous tensor field (practically, this may mean perturbations connected to some disease state) will still lead to
the formation of a non-ambiguous tensor field. The flexibility of DTT MRI allows making transformations from a problem with a non-unique solution to one with a unique solution by changing the acquisition schemes.

2.3. 3D case, secondary anisotropy $\lambda_1 > \lambda_2 > \lambda_3$

The existence of secondary myocardial anisotropy in the cross-fibre direction was previously established (Reese et al 1995). In this case it is necessary to estimate all three principal directions. These directions can be parametrized by three Euler’s angles, $\alpha(\vec{x})$, $\beta(\vec{x})$, and $\gamma(\vec{x})$, for each pixel. The corresponding lengthy notations are given in appendix B. Three different measurements (three different non-stationary $\vec{\omega}$) are required to reconstruct these three functions in DTT MRI. In the case of a stationary $\vec{\omega}$, as in conventional DT MRI, three measurements are not enough to obtain unique estimates of the principal directions. This can be understood in terms of 3D tensor invariants (Eisele and Mason 1970):

$$\text{Tr}(D) = D_{xx} + D_{yy} + D_{zz} = \lambda_1 + \lambda_2 + \lambda_3$$  \hspace{1cm} (9)

$$D_{xx}D_{yy} + D_{xx}D_{zz} + D_{yy}D_{zz} = \left(D_{xy}^2 + D_{xz}^2 + D_{yz}^2\right) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$  \hspace{1cm} (10)

and

$$\det(D) = D_{xx}D_{yy}D_{zz} + 2D_{xy}D_{xz}D_{yz} - \left(D_{xx}D_{yy}D_{zz} + D_{xz}D_{yz}^2 + D_{xy}D_{yz}^2\right) = \lambda_1\lambda_2\lambda_3.$$  \hspace{1cm} (11)

For example, if $D_{xx}$, $D_{yy}$ and $D_{zz}$ are given, $D_{xy}$ and $D_{xz}$ are defined up to the sign change. However, four measurements may be enough. The measurement of the components must be chosen carefully. It is a bad choice to measure $D_{xx}$, $D_{yy}$ and $D_{zz}$ among these four measurements. The tensor estimation will still be ambiguous.

3. Methods

3.1. Diffusion tensor phantom

Here we introduce a computer-generated phantom that was used to simulate the diffusion that can be expected in a cardiac study (see figure 3). The phantom comprises a cylindrical tube that simulates the mid-ventricular wall of the left ventricle. The spin density $\rho$ is assumed to be uniform inside the cylindrical annulus and zero elsewhere. The fibre structure of the myocardium is helical. The principal vectors of the diffusion tensor are referenced to a helical fibre structure with material coordinates $(X_R, X_F, X_C)$, which are orthogonal. The fibre axis $X_F$ is located on the plane of the wall normal to the radial axis $R$. The fibre angle in the circumferential direction has a variation that is continuous and linear. The angle changes from $60^\circ$ to $-60^\circ$, varying from the endocardial wall to the epicardial wall in a radial direction. The axis $X_C$ is the cross-fibre in-plane axis, and the axis $X_R$ coincides with $R$. The phantom was chosen to be independent of the $z$ coordinate so that one slice of the reconstruction was enough to represent the entire phantom.

We also used this 3D tensor field phantom to generate a 2D phantom, which is simply an axial view of the 3D phantom, with the exception that the vector field is 2D. In other words, the 2D phantom is similar to the 3D phantom where the $z$ component of the principal vectors is considered to be zero.

Various modifications of the 3D and 2D phantoms and the phantoms in figure 2 were used in computer simulations. In all cases the phantoms consisted of a $32 \times 32$ slice of a cylindrical cross-section with an inner radius $R_1 = 6$ and an outer radius $R_2 = 14$. This grid size was...
chosen in order to achieve a comprehensive visualization of the vector field of the principal vector, and to shorten reconstruction time. The eigenvalues of the myocardial diffusion tensor were taken from Reese et al (1995): $\lambda_1 = 1.6$, $\lambda_2 = 0.7$ and $\lambda_3 = 0.3$ in the 3D case and...
\[ \lambda_1 = 1.6 \text{ and } \lambda_2 = 0.7 \text{ in the 2D case.} \]

The unit dimension, which is not important here, was \(10^{-3} \text{ mm}^2 \text{s}^{-1}\). The spin density \(\rho\) was equal to 1 inside the cylindrical tube and zero elsewhere. The parameter \(b\) in equation (2) was 0.7, so \(b\lambda_1 > 1\).

Note that our 2D phantom in figure 3(b) is not an ambiguously defined tensor field for either the ‘plus’ or the ‘minus’ acquisition. As shown in appendix A for a vector \(\mathbf{\omega}\) that is parallel to \(\mathbf{\theta}\), a tensor field defined by an angle \(\Phi(\varphi, r) = \varphi + f(r)\) with \(f(r) = \pi/2\) does not define an ambiguous tensor field.

The projections were generated from pixelized versions of the phantoms using a ray-driven projection operator. Thirty-two projections of the slice were generated over \(\theta = [0, \pi)\). At each projection coordinate a scalar field along the line of projection was first calculated from the pixelized tensor field. Then the scalar field was used in the exponential term to evaluate the projection according to equation (2). In all cases the sampling bin width was equal to the reconstructed pixel width.

### 3.2. Reconstruction algorithm

Since the DTT MRI model in equation (2) is nonlinear, the reconstruction with non-stationary \(\mathbf{\omega}\) requires the use of iterative methods (Panin et al. 2000). To estimate the principal directions the least squares differences between modelled and measured projections \(y_{\mathbf{\omega}}(\theta, \xi)\) are minimized. In the 3D case:

\[
L(\alpha(\vec{x}), \beta(\vec{x}), \gamma(\vec{x})) = \sum_{\theta, \xi, \mathbf{\omega}} \| p_{\mathbf{\omega}}(\theta, \xi) - y_{\mathbf{\omega}}(\theta, \xi) \|^2 
\]  

and in the 2D case:

\[
L(\Phi(\vec{x})) = \sum_{\theta, \xi, \mathbf{\omega}} \| p_{\mathbf{\omega}}(\theta, \xi) - y_{\mathbf{\omega}}(\theta, \xi) \|^2.
\]  

In order to minimize the \(L\) norms in equations (12) and (13), with respect to the angular functions, a gradient-type algorithm was applied. Note that the minimization problem is complicated by the fact that the objective function is periodic with respect to the angular functions; therefore, \(L\) has an infinite number of minima. This is not a problem since the algorithm can converge to any minimum. The problem is that the \(L\) norm also has an infinite number of maxima, in which the gradient is zero and the algorithm may ‘stop’ or ‘be trapped’ at a maximum.

We have implemented two gradient-type algorithms. The first was the gradient descent (GD) algorithm to minimize \(L\). At each iteration this algorithm updates each angular function for a given pixel by its corresponding derivative of \(L\). This algorithm relies on an arbitrarily chosen relaxation parameter \(\varepsilon\), which defines the step size in the gradient direction. This parameter should be small enough to prevent over-shooting in the downhill direction. The choice of a relatively small value of \(\varepsilon\) leads to slow convergence. The second algorithm was the nonlinear Polak–Ribiere conjugate gradient (CG) algorithm with minimum bracketing (Press et al. 1997). The CG line search consisted of finding the zero of the gradient in the considered direction. Therefore, the CG algorithm can ‘be trapped’ around a local maximum even with minimum bracketing, which corresponds to the situation in which several pixels have values of angular functions that reach a maximum, not minimum, of \(L\).

We found that the CG algorithm converges much faster than the GD algorithm, as expected. On the other hand, the GD algorithm is more reliable, because it always points in the downhill direction. At the maximal points, where the gradient is almost zero, \(L\) is unstable when the GD algorithm is applied. Many iterations are required (when having small gradient values) to get out of the local maximum. The reconstructions below were obtained using the CG and
GD algorithms. In some cases, in order to determine if the result was a minimum, the CG reconstruction was used as a starting point to continue with the GD iteration process until convergence at a minimum was obtained.

The gradient of $L$, with respect to the angle $\Phi(x)$ at each pixel, was calculated in the projection and backprojection steps.

### 3.3. Figure of merit

To evaluate our results the difference in the angle $\delta$ between a principal vector $\mathbf{V}$ of the phantom and the principal vector $\mathbf{V}_{\text{recon}}$ of the reconstructed diffusion tensor for a given point was calculated according to:

$$\cos \delta = |\mathbf{V} \cdot \mathbf{V}_{\text{recon}}|.$$  \hspace{1cm} (14)

Note that $\delta$ has a positive value in the range $[0, \pi/2)$. This figure of merit was averaged over the entire phantom for each principal direction vector field.

### 4. Results

First, we considered the 2D case. Figure 4(a) shows the tensor fields that provide identical projection datasets when a stationary $\mathbf{\omega}(\phi = 0)$ is used. They all have the same $L$ norm, which is almost zero. Figure 4(b) represents the reconstruction from the uniform initial condition, $\Phi = \pi/4$, when a stationary $\mathbf{\omega}$ is used. The CG algorithm rapidly converges to an incorrect solution. These examples confirm our statement that the standard DT MRI leads to ambiguous principal directions on a point by point basis.

Figure 5(a) shows the reconstruction of ambiguous tensor fields from projections of figure 2((a) left) using the ‘plus’ acquisition. The tensor field from figure 2((a) right) was used as the starting point in the reconstructions from projections of the tensor field from figure 2((a) left). In theory these two tensor fields provide the same projection data. Figure 5(a) shows that the reconstruction is slightly noisy, but the error in the angle is about $60^\circ$, which is the true difference in principal direction angle between figures 2((a) right) and 2((a) left). The $L$ norm is much larger than expected since the projections of figures 2((a) right) and 2((a) left) are equivalent. This can be explained by the fact that our 2D phantom was generated on a grid and therefore is not perfectly rotationally symmetrical. The difference in the $L$ norms suggests that these discretized tensor fields are not exactly equivalent. (In standard DT MRI, discretization and these types of ambiguities are not an issue since tomography is not involved. Instead, the tensor field is reconstructed pixel by pixel directly from the same pixel by pixel sampling.) Note that the reconstruction from the uniform initial condition was also not close to the truth. The minimization problem has local minima that correspond to theoretically equivalent tensor fields. One may attempt to find the global minimum, which is likely to correspond to the true tensor field; however, it is better to create a problem without ambiguous solutions by using the ‘minus’ acquisition. In this case the algorithm converges very quickly (only 30 iterations). In our experience this convergence is independent of the initial condition. The results are presented in figure 5(b).

Figure 6 presents various modifications of the 2D phantom in figures 2 and 3(b). In the first example, figure 6(a) gives the reconstruction of projections of a phantom constructed by replacing a wedge in the upper right quadrant of the phantom from figure 2((a) left) with a wedge in the same location of the tensor field from figure 2((b) left). This perturbation does not change the performance of the reconstruction: when the ‘minus’ acquisition is used it is
Figure 4. Conventional PR DT MRI (axial views of the first principal directions). The projection data were generated from the 2D phantom from figure 3(b) and \( \phi = 0 \) was used. (a) Examples of tensor fields that produce equivalent projections of an \( L \) norm difference of approximately \( L = 10^{-8} \). (b) The reconstruction of the projections of the phantom in 3(b) using uniform initial conditions of \( \Phi = \pi / 4 \) for each pixel.
Figure 5. DTT MRI. Reconstruction of the tensor field from figure 2(a) left. (a) The ‘plus’ acquisition and the initial condition from figure 2(a) right were used. Note that the $L$ norm for this reconstruction is far from zero, which means that the reconstruction is far from the true tensor field. A histogram is plotted to demonstrate the distribution of the figure of merit for the reconstructed first principal directions. (b) The ‘minus’ acquisition and the same initial condition from figure 2(a) right were used. In other simulations it was observed that the same result was obtained using uniform initial conditions of $\Phi = \pi/4$ for each pixel.
Figure 6. Reconstruction of perturbed phantoms: (a) The reconstruction of projections of the phantom from figure 2(a) left) with the upper right quadrant replaced by the upper right quadrant of the phantom in figure 2(b) left). The ‘minus’ acquisition was used. (b) The structure of the 2D phantom from figure 3(b) was perturbed by noise with angular std of 18°. The ‘minus’ acquisition was used. (c) The reconstruction of the elliptical phantom. This tensor field can be reconstructed for ‘plus’ or ‘minus’ acquisitions.
easy to reconstruct the true tensor field, whereas, the reconstruction performance for the ‘plus’ acquisition is not as good, i.e. the algorithm becomes trapped in the local minimum.

In the second example, figure 6(b) presents a reconstruction of projections of the 2D phantom from figure 3(b) with the principal directions perturbed by noise, and with an angular standard deviation (std) of 18°. The nearly perfect reconstruction was achieved using the ‘minus’ acquisition (actually the reconstruction was correct for the ‘plus’ acquisition as well, since the original 2D phantom was well defined for both acquisitions). We can conclude that a well-defined tensor field is stable, even with significant perturbation, for any given acquisition type. In this case we also present the curves of the L norm as a function of the iteration number for the GD algorithm. As can be seen in comparison with the other examples, the GD algorithm converges much more slowly, but the curve of the L-norm versus the iteration number is smoother in comparison with the CG algorithm. On a PC with a Pentium 4 processor (1.7 MHz) 1000 iterations of the GD algorithm took 57 s (0.057 s/iteration) and the 100 iterations of the CG algorithm took 68 s (0.068 s/iteration).

The last example is a tensor field with an elliptical support in which the prototype was the phantom from figure 2((a) left) and the y-axis of the circular support was reduced by a factor of 2. The elliptical field was well reconstructed for the ‘plus’ and ‘minus’ acquisitions from various initial conditions. This example provides the hope that the ambiguously defined tensor fields in DTT MRI are limited to some rare examples with particular symmetrical properties.

In the following we consider 3D tensor fields. Figure 7 presents two examples of the tensor fields that provide the same projection datasets in PR DT MRI as the original 3D phantom in figure 3(a). Three stationary \( \vec{\omega} \) were used. Using these diffusion gradient directions, explicit reconstruction of \( D_{xx} \) and \( D_{yy} \) is obtained and \( D_{zz} \) can be obtained from the first invariant. The third \( \vec{\omega}_3 \), and knowledge of \( D_{xx} \) and \( D_{zz} \), provide enough information to obtain \( D_{xz} \). The other components of the tensor field are ambiguous in sign. This is a pixel by pixel ambiguity, as in the 2D case.

Figure 8 shows a DTT MRI reconstruction of the 3D phantom with the following three rotating diffusion gradient directions \( \vec{\omega} \):

\[
\vec{\omega}_1 = \begin{bmatrix} \phi = \pm \theta \\ \vartheta = \begin{cases} \theta & \theta \leq \pi/2 \\ \pi - \theta & \theta > \pi/2 \end{cases} \end{bmatrix}
\]

(15)

\[
\vec{\omega}_2 = \begin{bmatrix} \phi = \pm \theta + \pi/2 \\ \vartheta = \begin{cases} \pi/2 - \theta & \theta \leq \pi/2 \\ \theta - \pi/2 & \theta > \pi/2 \end{cases} \end{bmatrix}
\]

(16)

and

\[
\vec{\omega}_3 = \begin{bmatrix} \phi = \pm \theta + \pi/4 \\ \vartheta = \begin{cases} \theta + \pi/4 & \theta \leq \pi/4 \\ 3\pi/4 - \theta & \pi/4 \leq \theta \leq 3\pi/4 \\ \theta - 3\pi/4 & \theta > 3\pi/4 \end{cases} \end{bmatrix}
\]

(17)

which were chosen arbitrarily. The reconstruction for the ‘plus’ acquisition was obtained using uniform initial conditions where all three Euler angles were set to zero for each pixel. The reconstruction with 300 iterations of the CG algorithm is presented. Imperfections in the reconstruction, which were probably caused by the local maxima to which the CG algorithm could converge, were removed by further application of the GD algorithm to obtain a nearly perfect reconstructed tensor field.

For the 3D reconstruction (estimation of three Euler’s angles), the GD algorithm took 202 s for 1000 iterations (0.202 s/iteration) and the CG algorithm took 300 s for 100 iterations (3 s/iteration). These were performed on the same PC with a Pentium 4 processor (1.7 MHz) as in the 2D reconstructions cited above. In most cases, the CG algorithm converged faster
and thus required fewer iterations. However, the time per iteration was longer due to the fact that a line search occurred at each iteration, which was not the case for the GD algorithm.

Analogous to the 2D case, we also investigated a ‘minus’ acquisition, which corresponded to changing the angle $\theta$ in equations (15)–(17) to $-\theta$ for the rotation direction of the axial component of $\vec{\omega}$. We were able to reconstruct the original tensor field for both the ‘plus’ and ‘minus’ acquisitions independent of the initial conditions. The projections of two vectors $\vec{\omega}_1$ and $\vec{\omega}_2$ into axial plane are mutually orthogonal but projections of all three omega vectors ($\vec{\omega}_1$, $\vec{\omega}_2$, $\vec{\omega}_3$) are not mutually orthogonal. Three vectors are enough to provide a unique reconstruction in the axial plane; therefore the analogy to the 2D ambiguously defined tensor fields does not apply here. In fact, we have not yet been able to construct an example of an ambiguous tensor field, but their existence appears to be likely based on our 2D experience.

5. Discussion

A novel approach to reconstructing the principal directions of the diffusion tensor field was developed. It was demonstrated that DTT MRI data acquisition is more efficient and more
flexible than standard DT MRI data acquisition. However, the choice of an optimal direction of the diffusion gradient during acquisition still needs further investigation.

For 2D DT MRI it was demonstrated that one set of projections is insufficient for reconstructing the principal vectors. However, with DTT MRI one set of projections with rotating diffusion gradients along \( \vec{\omega} \) is sufficient for reconstructing the principal vectors, except for some rare tensor fields where an improper \( \vec{\omega} \)-rotation scheme can lead to ambiguity. The ambiguity disappears for the same tensor fields when a different \( \vec{\omega} \)-rotation scheme is applied.

Similarly, in 3D DT MRI three sets of projections are insufficient for reconstructing the principal vectors; however, in DTT MRI three sets of projections of rotating diffusion gradients defined by three independent directional vectors \( \vec{\omega}_1, \vec{\omega}_2 \) and \( \vec{\omega}_3 \) are sufficient. In computer simulations for the case of \( \lambda_1 > \lambda_2 > \lambda_3 \), no ambiguous reconstructions were obtained using three independent sets of projections. We surmise that ambiguity in the reconstruction may occur in a very small number of tensor fields with rare symmetrical properties and for a
particular $\vec{\omega}_1$, $\vec{\omega}_2$ and $\vec{\omega}_3$ rotating scheme, even though we have not found such an example yet.

In normal cases, as in the heart, the eigenvalues will be known, whereas in cases of disease the eigenvalues will not be known. In some disease conditions there is loss of helical fibre structure. For example, in hypertrophic cardiomyopathy, there may be a disordered presence of various-sized interlacing hypertropic fibres with swirling patterns that disrupt the helical fibre structure (Farrer-Brown 1977). One must watch for disease states that disrupt the structure. In practice, the reconstructed tensor field can first be investigated; it is easy to recognize an image that no longer has a normal helical fibre structure. This also can be aided by a segmentation algorithm. In those cases it is important to modify the eigenvalue in regions of disease and reconstruct the tensor field again. It is not entirely clear at this point how robust our method will be if the wrong eigenvalue is selected. More computer simulations need to be performed to evaluate this. It is conceivable that an algorithm that alternates between the estimation of the eigenvector direction and the estimation of the eigenvalue with appropriate constraints might be developed. In general, we believe tomography will be useful in reducing the number of measurements, or improving the signal-to-noise ratio if the same number of measurements are employed.

It is expected that the algorithm will be readily applicable to real data. However, there is one caveat, and that is, possible negative effects may occur when structured noise is present. The signal-to-noise ratio in MRI is very good, and in our simulations, using the level of noise in MRI (signal-to-noise ratio of 30), we obtain accurate results. For the case of the 2D phantom of figure 5(b), using ‘minus’ acquisition the figure of merit changed from $\langle \delta \rangle = 0.9^\circ$ (noise-free) to $\langle \delta \rangle = 5.6^\circ$ (with noise). However, in working with experimental data we find more problems with structured noise caused by table vibration, unstable gradients and eddy currents that vary with the rotation of high amplitude diffusion gradients. These can result in severe artifacts in the reconstructed tensor fields. Our experience to date is that the rotation of high-amplitude diffusion gradients is not stable in several MRI systems. The hardware must be made more stable before DTT MRI can be made a clinically useful tool.

It is well recognized that DT MRI is a valuable diagnostic tool in clinical medicine. It may be that DT MRI will eventually replace histology procedures, which are very costly and time consuming (Scollan et al 1998). DTT MRI offers the potential to utilize MRI in a more efficient way for some specific applications.

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Appendix A

Here we construct an example of an ambiguously defined tensor field in 2D DTT MRI for an acquisition utilizing the simple linear relationship between vectors $\vec{\omega}$ and $\vec{\theta}$: $\phi = \pm \theta$, where the plus sign is used for the ‘plus’ acquisition and the minus sign for the ‘minus’ acquisition.

Let the 2D spin density function be:

$$\rho(x) = \begin{cases} 
\rho_0 & \text{if } r_1 \leq \sqrt{x^2 + y^2} \leq r_2 \\
0 & \text{otherwise} 
\end{cases}$$

(A1)
If the tensor ‘dot product’ is rewritten as
\[
\vec{\omega}^T (\phi = \pm \theta) D(\vec{x}) \, \vec{\omega}(\phi = \pm \theta) = \frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2} \cos 2(\mp \theta + \Phi(\vec{x}))
\] (A2)
then the diffusion-weighted spin density function can be expressed as
\[
F(\cos 2(\mp \theta + \Phi(\vec{x}))) = \rho(\vec{x}) \exp(-b \vec{\omega}^T (\phi = \pm \theta) D(\vec{x}) \, \vec{\omega}(\phi = \pm \theta)).
\] (A3)

The projection from this function is:
\[
p_\theta(\xi) = \int F(\cos 2(\pm \theta + \Phi(\vec{x}))) \delta(\vec{x} \cdot \vec{\theta} - \xi) \, d\vec{x}.
\] (A4)

Now consider the tensor field with ‘rotational’ symmetry in polar coordinates \((r, \phi)\) (see figure A1) \(\Phi(\vec{x}) = \pm \phi + f(r)\) where \(f(r)\) is a certain function. The polar angle can be rewritten in rotated system coordinates \((\xi, \zeta)\) as \(\phi = \theta + \arctan(\zeta/\xi)\), then
\[
p_\theta(\xi, \zeta) = \int_{-\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}}^{\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}} F \left( \cos 2 \left( \pm \arctan \frac{\zeta}{\xi} + f \left( \sqrt{\xi^2 + \zeta^2} \right) \right) \right) \, d\zeta
\]
\[+ \int_{-\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}}^{\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}} F \left( \cos 2 \left( \pm \arctan \frac{\zeta}{\xi} + f \left( \sqrt{\xi^2 + \zeta^2} \right) \right) \right) \, d\zeta
\]
\[= \int_{\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}}^{\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}} F \left( \cos 2 \left( \pm \arctan \frac{\zeta}{\xi} - f \left( \sqrt{\xi^2 + \zeta^2} \right) \right) \right) \, d\zeta
\]
\[+ \int_{\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}}^{\sqrt{\frac{\zeta^2 - \xi^2}{\xi^2 - \zeta^2}}} F \left( \cos 2 \left( \pm \arctan \frac{\zeta}{\xi} + f \left( \sqrt{\xi^2 + \zeta^2} \right) \right) \right) \, d\zeta.
\] (A5)

It is clear that the function \(f\) can be defined up to the sign, that is, the projection is the same for each projection view when \(f\) is replaced by \(-f\). Any diffusion-weighted spin density function combined from such rings leads to ambiguously defined tensor fields for particular DTT MRI data acquisition schemes.
Note that in order to construct this particular example of ambiguity, we required that the spin density be uniform. While the eigenvalues might be the same for any particular tissue type, and the parameter \( b \) may be constant over the imaging region because of the large value of the diffusion gradients, the spin density function can be non-uniform for the tissue because of variations in sensitivity along a projection profile. This may destroy the symmetry and eliminate the ambiguity.

Appendix B

The 3D rotation is parametrized by three Euler’s angles \( \alpha, \beta \) and \( \gamma \), see figure B1. The three basis vectors \( V_1 = [1, 0, 0] \), \( V_2 = [0, 1, 0] \) and \( V_3 = [0, 0, 1] \) have the following coordinates after a 3D rotation:

\[
V_1 = [c_2c_3 - c_1s_2s_3, s_2c_3 + c_1c_2s_3, s_1s_3]
\]  
\[
V_2 = [-c_2s_3 - c_1s_2c_3, -s_2s_3 + c_1c_2c_3, s_1c_3]
\]

and

\[
V_3 = [s_1s_2, -s_1c_2, c_1]
\]

where \( c_1 = \cos \alpha, c_2 = \cos \beta \) and \( c_3 = \cos \gamma \); and \( s_1 = \sin \alpha, s_2 = \sin \beta \) and \( s_3 = \sin \gamma \).

The diffusion tensor is both

\[
D_{ij} = \lambda_1 V_{1,i}V_{1,j} + \lambda_2 V_{2,i}V_{2,j} + \lambda_3 V_{3,i}V_{3,j}
\]

and

\[
\vec{\omega}^T D \vec{\omega} = \lambda_1 (\vec{\omega}^T V_1)^2 + \lambda_2 (\vec{\omega}^T V_2)^2 + \lambda_3 (\vec{\omega}^T V_3)^2.
\]

The cardiac phantom from figure 3(a) can be parametrized by the following set of Euler’s angles (in the polar coordinate system of one slice, see appendix A):
\[
\alpha(r, \varphi) = \Psi_0 \frac{R_2 - r}{R_2 - R_1} - \Psi_0 \frac{r - R_1}{R_2 - R_1} \quad \Psi_0 = \frac{\pi}{3} \quad (B6)
\]
\[
\beta(r, \varphi) = \varphi \quad (B7)
\]
\[
\gamma(r, \varphi) = \frac{\pi}{2} \quad (B8)
\]

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