Efficient cardiac diffusion tensor MRI by three-dimensional reconstruction of solenoidal tensor fields

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Abstract

Tensor tomography is being investigated as a technique for reconstruction of in vivo diffusion tensor fields that can potentially be used to reduce the number of magnetic resonance imaging (MRI) measurements. Specifically, assessments are being made of the reconstruction of cardiac diffusion tensor fields from 3D Radon planar projections using a filtered backprojection algorithm in order to specify the helical fiber structure of myocardial tissue. Helmholtz type decomposition is proposed for 3D second order tensor fields. Using this decomposition a Fourier projection theorem is formulated in terms of the solenoidal and irrotational components of the tensor field. From the Fourier projection theorem, two sets of Radon directional measurements, one that reconstructs the solenoidal component and one that reconstructs the irrotational component of the tensor field, are prescribed. Based on these observations filtered backprojection reconstruction formulae are given for the reconstruction of a 3D second order tensor field and its solenoidal and irrotational components from Radon projection measurements. Computer simulations demonstrate the validity of the mathematical formulations and demonstrate that a realistic model of the helical fiber structure of the myocardial tissue specifies a diffusion tensor field for which the first principal vector (the vector associated with the maximum eigenvalue) of the solenoidal component accurately approximates the first principal vector of the diffusion tensor. A priori knowledge of this allows the orientation of the myocardial fiber structure to be specified utilizing one half of the number of MRI measurements of a normal diffusion tensor field study. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

There is an increasing need to measure in vivo tensor quantities (diffusion, strain, stress, and conductivity) for the purpose of developing more accurate models of the properties of biological tissue in order to enable better diagnosis of various diseases. Magnetic resonance imaging (MRI) has been shown to be effective for imaging brain [1] and cardiac [2–5] diffusion tensor fields, and cardiac strain-rate tensor fields [6–8]. In particular, diffusion-tensor MRI may eventually be useful for characterizing myocardial fiber structure [3]. Knowledge of fiber bundle orientation will be useful for specifying material axes for mechanical models [9–14] and for identifying conductive pathways for electrical models of the heart [15]. We point out that all of this work involves the use of non-tomographic techniques to measure diffusion and strain-rate tensor fields. However, it may be advantageous to use MRI tensor computed tomography, which in certain applications may provide information in fewer measurements and may provide images that are not compromised by artifacts due to motion or eddy currents. The aim of this paper is to develop three-dimensional inversion formulae for the reconstruction of second order tensor fields from Radon planar projections and to illustrate, through computer simulations, that diffusion imaging of myocardial fibers may be performed with fewer measurements, thereby reducing imaging time.

First, we must make it clear that tensor tomography is not necessary for obtaining 3D maps of in vivo tensor fields. MRI is unique in that it does not require computed tomography to form a three-dimensional image of internal structures. However, recently there has been much interest in using projection reconstruction techniques for various MRI applications because projection reconstruction techniques are useful for processing rapidly acquired data which can be used to follow dynamic processes and may be less sensitive...
to motion and eddy currents. Of recent interest is the use of projection reconstruction techniques to image diffusion tensor fields [16]. The tomographic method that is used is quite different than the method that we describe in this paper. Diffusion weighted images are obtained from projection reconstruction techniques. From the reconstructed diffusion weighted images the tensor fields are calculated using standard methods of combining appropriate gradient weighted reconstructed images [17,18]. The method we propose differs in that scalar projections of the diffusion tensor field are obtained at several orientations and the diffusion tensor field, instead of a diffusion-weighted image, is reconstructed directly from these projections [19].

This brings us to an important second point. At present, there is no known method to form scalar projection measurements of tensor fields without performing some approximation. That is, there is no existing tensor detector. We point out that in our method, to form projections of diffusion tensor fields, an approximation of the MRI signal had to be made [19]. This approximation was similar to that proposed by Prince [20] in using MRI to form projections of flow velocity vector fields. A past problem in the application of vector field tomography was not so much the formation of scalar projections but the forming of these scalar projections from arbitrary directions. As in ultrasound time of flight measurements, techniques did not exist to measure all of the necessary components of the vector field. Only the component of the vector field projected onto the line of integration (longitudinal measurements) could be measured [21], which was not sufficient for reconstructing the complete vector field. MRI on the other hand can form measurements from arbitrary directions. An approximation still must be made to form each measurement as a scalar projection.

The development of vector and tensor field tomography has relied heavily on the Helmholtz decomposition of a vector field from which we derive the focus of this paper. The Helmholtz decomposition states that if the source and circulation components of a vector field vanish at infinity that field can be written in a unique way as the sum of a solenoidal component (divergence-free, also referred to as source-free component) and an irrotational component (curl-free component) [22]. The solenoidal component can be expressed as the curl of a vector potential \( \nabla \times \psi \) and the irrotational component can be expressed as the gradient of a scalar potential \( \nabla \Phi \). In our discussion to follow, one will appreciate the importance of the Helmholtz decomposition in developing relationships between projection measurements that reconstruct elements of the solenoidal component of the vector field and projection measurements that reconstruct elements of the irrotational component of the vector field.

Tensor tomography builds on much of the work that has already been accomplished in vector field tomography [20,21,23–49]. Sparr et al. [49] give an excellent review of this work. Several of the applications have involved acoustic flow imaging using time-of-flight measurements—ultra-sonic imaging in medicine [21], acoustic flow imaging in nondestructive evaluation [23], and ocean acoustic tomography [24,25]. Johnson et al. [21] performed one of the earliest vector field tomographic studies using ultrasound to reconstruct velocity vector fields in blood vessels from acoustic time-of-flight measurements. The projection measurements were the integral of the projected component of the velocity vector field onto the line forming the projection measurement (also called the longitudinal projection measurement [32]). An iterative algebraic reconstruction algorithm (ART) was used to compute what Norton [26,27] pointed out later to be the divergence-free component of the vector field fulfilling homogeneous Neumann boundary conditions (i.e., the normal component of the divergence-free component at the boundary is zero). Using the Helmholtz decomposition and the Fourier central-slice theorem, Norton [26,27] derived a reconstruction method for the velocity field and showed that the reconstruction of the acoustic time-of-flight (longitudinal projection) measurements with boundary conditions allows reconstruction of a divergence-free vector field composed of a solenoidal component that satisfies homogeneous boundary conditions and an irrotational component defined by the gradient on the boundary. It was shown that the longitudinal measurements alone could not recover the irrotational component of the vector field. Additional boundary condition information was needed to reconstruct the irrotational component of a divergence-free vector field.

Before Norton’s important contribution, Kramer and Lauterbur [28] developed a hybrid filtered backprojection algorithm for the reconstruction of flow using nuclear magnetic resonance (NMR). They showed that some flow components could not be reconstructed from the longitudinal projection measurements. Working independently, Winters and Rouseff [29] also developed a Fourier central section theorem for the reconstruction of the divergence-free component, which they argued was important for specifying the vorticity of fluid flow [29,30,31].

Later Braun and Hauck [32] showed (not using Fourier projection techniques but spatial convolution techniques) that projection of the orthogonal component of the velocity vector field (the transverse projection measurement, which is the integral of the orthogonal component of the velocity vector field along the line of the projection measurement) allows one to reconstruct the irrotational component of the vector field. This irrotational (curl-free) component along with the divergence-free (solenoidal) component gives the complete determination of the flow vector field. Braun and Hauck [32] recognized that bounded domains admit harmonic vector fields that are both irrotational and solenoidal. Therefore, the decomposition into irrotational and solenoidal components is not unique. In their paper, they proposed that the decomposition should be \( V = V_{\psi}^S + V_{\Phi}^I + V_H^I \), where \( V_{\psi}^S = \nabla \times \Psi^S \), \( V_{\Phi}^I = \nabla \Phi^I \), and \( V_H^I \) is the harmonic component of the vector field satisfying \( \nabla \cdot V_H^I = 0 \) and \( \nabla \times V_H^I = 0 \). The solenoidal component \( V_{\psi}^S \) is
homogeneous in the sense that the normal component of \( V_\Psi^s \) is zero on the boundary and \( V_\Psi^s \) is completely tangential to the boundary. The curl-free component \( V_\Phi^t \) is homogenous in the sense that the tangential component of \( V_\Phi^t \) vanishes on the boundary and is exactly normal to the boundary.

Later, Prince [20] extended the previous work in 2D to 3D by developing filtered backprojection algorithms that could be used to reconstruct both the solenoidal and irrotational components of the vector field from 3D Radon projections. Prince generalized the longitudinal and transverse measurements of Braun and Hauck [32] by defining a more general inner product measurement (probe transform) that forms an inner product between the vector field and a unit-vector probe direction. Prince [20] showed that in 3D only one set of probe measurements (the irrotational measurements) is required to reconstruct the irrotational component and that two sets of probe measurements (the solenoidal measurements) are required to reconstruct the solenoidal component. The three probe directions form a linearly independent set of vectors. (The study by Prince [20] is particularly noteworthy because it defines the principles from which methods can be developed for measuring projections of diffusion tensors using MRI [19].) Sparr [49] took this further and showed that the vector ray transform (also referred to as the vector X-ray transform) can recover the solenoidal component of the vector field and the vector Radon normal transform can recover the irrotational component of the vector field. However, these results need to be studied further because it is not clear what type of angular sampling on the unit sphere is necessary to recover the solenoidal component. The question remains whether it is necessary to sample the entire unit sphere, or if sampling over one great circle is sufficient, or whether sampling over multiple great circles is required to reconstruct the solenoidal component.

Norton [26–27], and Braun and Hauck [32], and later Osman and Prince [33] were all concerned about the vector tomography problem on bounded domains. In fact, physical problems are often defined on bounded domains and it is the boundary that creates or partially defines the field. Braun and Hauck [32] considered the 2D problem on a circular domain and later Osman and Prince [33] extended that work to a general 3D domain. Braun and Hauck [32] showed that the harmonic component is reconstructed equally between the irrotational and solenoidal measurements. Whereas, Osman and Prince [33] showed that the harmonic component of a 3D vector field is not imaged equally in the irrotational and solenoidal measurements. In fact Osman and Prince [33] characterized the homogenous component as \( V_\Psi^s \) where \( V_\Psi^s = -\nabla \Xi \) for some harmonic function \( \Xi \) satisfying Laplace’s equation \( \nabla^2 \Xi = 0 \) on the domain. The work of Osman and Prince [33] considered only the Radon transform. They showed that the irrotational measurements could be used to reconstruct both the irrotational component, which has homogeneous boundary conditions, and the harmonic component that arises from the normal field component on the boundary. By solving the Poisson equation \( \nabla^2 \Phi = \nabla^2 \Phi \) \( (\Phi = \Phi_I + \Phi_H \) is known from the irrotational field reconstruction) with the Dirichlet boundary conditions \( \Phi_I = C \), one can separate the homogeneous and harmonic components of the irrotational component. Similarly, the solenoidal measurements can be used to reconstruct both the solenoidal component, which has homogeneous boundary conditions, and the harmonic component that arises from the tangential field component on the boundary. By solving \( \nabla \times \nabla \times \Psi = \nabla \times \nabla \times \Psi \) \( (\Psi = \Psi_S + \Psi_H \) is known from the solenoidal field reconstruction) with the solenoidal boundary conditions \( \nabla \times \Psi \) is tangential on the boundary), one can separate homogeneous and harmonic components.

Works of vector tomography have been applied in other applications such as: optics, to measure flow [26,27]; deflection optical tomography, to determine densities in supersonic expansions and flames [34]; optical tensor field tomography, to determine stress in photoelastic materials by measuring polarization of the transmitted light [35–39]; and acoustics, to determine 3D temperature and velocity fields in furnaces [40] and velocity fields of heavy particles in plasma [41]. In other works, it has been shown that continuous Doppler data can also be analyzed in the framework of vector tomography [27,42–46,49].

As seen in the previous discussion, vector field tomography has been an active area of research for several applications including medical imaging. With all of this activity though, there has been little work in developing algorithms for the tomographic reconstruction of tensor fields, which has great potential for application in medical imaging. A National Science Report [50] and the review by Sparr et al. [49] alluded to the potential for tensor field tomography. Recently, we presented [19] the results of computer simulations and results of an MRI diffusion experiment in which tensor tomography was applied in 2D. Sharafutdinov [51] presented a monograph on integral geometry of general n-th order tensor fields—wherein the problem of reconstructing a symmetric tensor field from its integrals over all geodesics of a given Riemannian manifold is solved. A most interesting result presented in this monograph is that a symmetric tensor field has a Helmholtz-type decomposition in that it can be decomposed into a sum of a divergence-free component and a curl-free component. In this paper we use this result to decompose a 3D second order tensor field into a solenoidal component and an irrotational component that can be expressed as the gradient of a vector potential. This result is analogous to the Helmholtz decomposition of a vector field. The important aspect of this decomposition is that if the tensor field is recognized to be solenoidal or irrotational it can be reconstructed from fewer measurements thereby requiring less time to acquire the data.

The principles of scalar tomography can fairly easily be extended to tensor tomography. Though only second
order tensors are considered in this paper the theory can be extended to \(n\)th-order tensors as well [51] (also see [52–55] for texts on tensors analysis). The scalar projection measurements of the tensor field are formed by creating inner products that are the contraction of the tensor with unit directional vectors to give a scalar quantity at each point along the line of the projection measurement. The formation of these projections from MRI measurements is presented in the Appendix of this paper. Both analytical and iterative reconstruction algorithms can be used to reconstruct tensor fields. Iterative algorithms can be developed for the reconstruction of 3D tensor fields using the same techniques as those previously implemented for the reconstruction of 2D vector fields [21,40,47,48]. However, we only consider an analytical approach in this paper. Analytical algorithms are appealing because they lend themselves better to a geometric interpretation of the physical problem. Iterative algorithms are appealing because they can perform better reconstructions when the data are limited or noisy. Also, in order to reconstruct diffusion tensor fields without any approximation of the MRI measurements, an iterative algorithm is necessary to be able to solve the nonlinear projection equation [56].

In our initial work a reconstruction algorithm was developed for the reconstruction of 2D second order tensor fields [19]. In that study, Fourier projection theorems were developed for the solenoidal and irrotational components of a 2 \(\times\) 2 second order tensor. These algorithms were verified in computer simulations and applied to the reconstruction of two-dimensional diffusion tensor fields from MRI tensor projection measurements.

Here we extend our development of 2D reconstruction of tensor fields to the first development of 3D Radon tensor tomography. First 3D Radon planar projections of tensor fields are defined, then a Helmholtz-type decomposition is presented for tensor fields. Using this decomposition a Fourier projection theorem is formulated in terms of the solenoidal and irrotational components of the tensor field. From that, it is shown that the solenoidal and irrotational components of the tensor field can each be reconstructed from a set of three Radon directional measurements using a filtered backprojection algorithm. Computer simulations were used to demonstrate the reconstruction of diffusion tensor fields defined in a cylindrical phantom intended to simulate the mid-ventricular region of the heart. A simulation of the cardiac diffusion tensor field was studied since it correlates with the helical fiber structure of myocardial tissue. It is shown that this structure specifies a diffusion tensor field with a first principal vector that is nearly equal to the first principal vector of the solenoidal component of the tensor field. A priori knowledge of this allows this principal vector to be specified in one half of the number of MRI diffusion tensor measurements of a normal diffusion tensor field study.

2. Theory

A tensor field in \(\mathbb{R}^3\) will be denoted by its nine elements \(t_{ij}(x), \ i, j = 1, 2, 3\) where \(x = (x, y, z) \in \mathbb{R}^3\). Each element \(t_{ij}(x)\) of the tensor field is a real, rapidly decreasing \(C^\infty\) function defined on \(\mathbb{R}^3\). In this paper we only consider tensor fields defined in the Euclidean space \(\mathbb{R}^3\).

2.1. Directional Radon projection measurements of tensor fields

The 3D Radon transform is normally defined for scalar functions. Projection imaging of a tensor field involves measuring the usual 3D Radon transform of the scalar product of the tensor field with a pair of unit vectors, thus defining a scalar directional measurement. In reference to the directional 3D Radon transform of the tensor field \(t_{ij}(x)\) is defined by

\[
\rho^{\lambda\tau}(t; \theta) = \int_{\mathbb{R}^3} \sum_{i,j} e^{i\lambda t_{ij}(x)} e^{i\tau(x \cdot \theta - t)} \, dx
\]

For \(\lambda, \tau = 1, 2, 3\).

(1)

The tensor is contracted with two unit vectors \(\mathbf{e}_\lambda\) and \(\mathbf{e}_\tau\) to obtain a scalar function that is integrated over a plane. This produces directional projection measurements defined by the three-dimensional directional unit vectors \(\mathbf{e}_\lambda\) and \(\mathbf{e}_\tau\). We use a shortened notation \(\rho^{\lambda\tau}\) instead of \(\rho^{\lambda\tau}(t; \theta)\) to identify the directional projections defined by the vectors \(\mathbf{e}_\lambda\) and \(\mathbf{e}_\tau\). We also use the notation \(\rho^{\lambda\tau}(t; \theta)\) to identify the directional projections defined by the vectors \(\mathbf{e}_\lambda\) and \(\mathbf{e}_\tau\). The notation will
become clear from the context in which it is used. It is an interesting subject to consider, but at this point we make no statement as to whether \( r^\lambda_\tau \) is also a tensor.

If we consider only symmetric tensors it is obvious that six independent directional measurements are required to recover the 3D tensor field \( t_{ij} \), \( i, j = 1, 2, 3 \). We could for instance use unit vectors \( e^1 = (1, 0, 0) \), \( e^2 = (0, 1, 0) \), and \( e^3 = (0, 0, 1) \) along the three axes, which would amount to separately measuring the Radon transform of the six elements of the tensor (that is, the element-wise Radon transform) by measuring the pairs \( r^\lambda_\tau \) with \( (\lambda, \tau) = (i, i), (j, j), (k, k), (i, j), (i, k), \) and \( (j, k) \). Each element would then be reconstructed independently using the usual inverse 3D Radon transform.

It is often more natural to consider general sets of directional data defined using unit vectors which may depend on the direction \( \theta \) normal to the plane that defines the planar projection defined in Eq. (1) and illustrated in Fig. 1. Here we will restrict ourselves to the following system of orthogonal vectors

\[
e^1 = \theta = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \tag{2}
\]

\[
e^2 = \alpha = (-\sin \phi, \cos \phi, 0), \tag{3}
\]

\[
e^3 = \beta = (-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta). \tag{4}
\]

The scalar directional measurements of vector fields involve the dot product of the vector field with one directional unit vector. In 2D, Braun and Hauck [32] considered transverse and longitudinal projection measurements and identified them to correspond to the use of directional unit vectors \( \theta \) and \( \hat{\theta} \), orthogonal to \( \theta \), respectively. Prince [20] generalized the concept of directional measurements in 3D and referred to them as probe measurements. Prince showed that a 3D vector field can be recovered from the 3D Radon transform of the vector field if directional measurements are formed for any arbitrary set of vectors \( e^1(t, \theta), e^2(t, \theta), \) and \( e^3(t, \hat{\theta}) \), which are linearly independent at each \( (t, \theta) \). The unit directional vectors in Eq. (2)–Eq. (4) are just one possible set of directional vectors.

2.2. Projection theorem for directional Radon projections

The projection theorem (central section theorem) for scalar tomography is easily extended to a projection theorem for directional Radon projections of a tensor field. If the directional Radon projection of a tensor field is written as it is in Eq. (1) then the central section theorem follows from the linearity of the tensor operations and the standard scalar central section theorem. This leads to the following formulation of the central section theorem for the directional Radon transform of a tensor field:

\[
F^\lambda_\tau(v_i; \theta) = \sum_{i,j} e^i e^j F_t(t_i, t_j)(v_i, \theta) e^j \quad v_i \in \mathbb{R}
\]

\[
\lambda, \tau = 1, 2, 3. \tag{5}
\]

Here we assume that the Fourier transform \( \mathcal{F} \) of \( f(t) \) is defined as \( \mathcal{F}(f)(v) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i v t} dt \). Therefore, the Fourier transform of a directional Radon projection is the scalar product of the Fourier transform of the tensor field with the pair of directional unit vectors \( e^\lambda \) and \( e^\tau \) defining the directional projection. It is also immediately obvious that the Fourier transform \( \tilde{t}_{ij} \) of the tensor elements is also related to the Fourier transform of the projection of the individual elements of the tensor field. This will become useful later in deriving an inverse relationship.

2.3. Tensor field decomposition

It was shown by Sharafutdinov [51] that a smooth symmetric tensor field, which vanishes rapidly enough at infinity, could be decomposed in a unique way as

\[
t_{ij}(x) = t^\lambda_\tau(x) + \frac{1}{2} (\partial_i \phi_j(x) + \partial_j \phi_i(x)), \tag{6}
\]

where \( \phi(x) \) is a vector potential specifying the irrotational component of the tensor field and \( t^\lambda_\tau \) is a symmetric solenoidal tensor field that is divergence free:

\[
\sum_i \partial_i t^\lambda_\tau(x) = \sum_j \partial_j t^\lambda_\tau(x) = 0,
\]

where \( \partial_x t^\lambda_\tau = \partial^\lambda \partial_x \partial_j \). This is a generalization of the well-known vector field decomposition described by Helmholtz [22]. Note, however, that it says nothing about the solenoidal component being a curl of a tensor potential, which one might infer as an extension of the Helmholtz vector field decomposition to tensor fields [19].

Since the partial derivative, with respect to \( x_j \), is equivalent to the multiplication of the Fourier transform by \( 2\pi i v_j \), we obtain the following for the Fourier transform of the decomposition of the tensor field

\[
\tilde{t}_{ij}(v) = \tilde{t}^\lambda_\tau(v) + \tilde{t}^\lambda_\tau(v) \quad v \in \mathbb{R}^3 \tag{8}
\]

where

\[
\tilde{t}^\lambda_\tau(v) = 2\pi i \left[ \frac{1}{2} v_i \partial_j \phi_j(v) + \frac{1}{2} v_j \partial_i \phi_j(v) \right]. \tag{9}
\]

Note that the irrotational component of the tensor field depends upon three scalar functions \( \phi_1, \phi_2, \) and \( \phi_3 \). Taking the Fourier transform of Eq. (7) yields

\[
\sum_i v_i \tilde{t}^\lambda_\tau(v) = \sum_j v_j \tilde{t}^\lambda_\tau(v) = 0, \tag{10}
\]

which indicates that in Fourier space the solenoidal component is orthogonal to the frequency coordinate \( \mathcal{E} \). In contrast, from Eq. (9), we see that the irrotational component is parallel to \( \mathcal{E} \) in Fourier space. Therefore, another way to interpret the decomposition is by separating the components, in Fourier space, that are orthogonal and parallel to the frequency \( \mathcal{E} \). The solenoidal component is then given by simply projecting the Fourier transform of the tensor field onto the plane orthogonal to \( \mathcal{E} \) [51]:

\[
\tilde{t}^\lambda_\tau(v) = \mathcal{P}_\mathcal{E} \tilde{t}^\lambda_\tau(v) = \sum_j v_j \tilde{t}^\lambda_\tau(v) = 0, \tag{11}
\]

where \( \mathcal{P}_\mathcal{E} \) is the projection operator onto the plane orthogonal to \( \mathcal{E} \).
\[ t_0^j(v) = \tilde{t}_0^j(v) - n \left[ \sum_k n_k \tilde{t}_k(v) \right] - n \left[ \sum_k n_k \tilde{t}_k(v) \right] + n \rho_j \left[ \sum_k n_k \rho_j \tilde{t}_k(v) \right], \]

(11)

where \( \rho = \| \tilde{v} \|_2 \).

Expressing the central section theorem for the directional Radon transform in terms of the decomposition [using Eq. (8) and Eq. (9)]:

\[ t_\theta^j(v) = \tilde{t}_\theta^j(v) + 2\pi i \left[ \frac{1}{2} \tilde{\phi}_i(v) + \frac{1}{2} \nu \tilde{\phi}_i(v) \right], \]

(12)

we have

\[ \tilde{t}_\theta^j(v) = \sum_{i,j} e_i \tilde{t}_\theta^j(v_{ij}) e_j \]

\[ + 2\pi i \nu \sum_{i,j} e_i \left[ \frac{1}{2} \theta \tilde{\phi}_i(v_{ij}) \right] e_j, \]

(13)

where \( \theta_i \) are elements of the directional vector \( e^1 = \theta \).

Symbolically, we write the Fourier transform of the directional Radon projections in Eq. (13) as

\[ \tilde{R}(v; \theta) = \begin{bmatrix} \tilde{r}^{00} & \tilde{r}^{0a} & \tilde{r}^{0b} \\ \tilde{r}^{a0} & \tilde{r}^{aa} & \tilde{r}^{ab} \\ \tilde{r}^{b0} & \tilde{r}^{ba} & \tilde{r}^{bb} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{l}_1 & \tilde{l}_2 \\ 0 & \tilde{l}_2 & \tilde{l}_3 \end{bmatrix} + \begin{bmatrix} 1 \tilde{I}_1 \tilde{I}_2 \tilde{I}_3 \\ 0 \tilde{I}_2 \tilde{I}_3 \\ 0 \tilde{I}_3 \end{bmatrix} \]

(14)

where all matrix elements are a function of \((v; \theta)\). The first matrix after the second equality consists of the solenoidal component in Eq. (13) and the second matrix consists of the irrotational component. The elements of the solenoidal component that are zero follow from the fact that the solenoidal decomposition is divergence free [Eq. (10)]. The elements of the irrotational component in Eq. (13) and the second matrix consists of the irrotational component. The elements of the solenoidal component in Eq. (13) and the second matrix consists of the irrotational component. The elements of the solenoidal component that are zero follow from the fact that the solenoidal decomposition is divergence free [Eq. (10)]. The elements of the irrotational component in Eq. (13) and the second matrix consists of the irrotational component.

Solving the matrix equation in Eq. (18) for \( \tilde{I} \), we have

\[ \tilde{I}_3 = \pi i v \left[ -\cos(\theta) \cos(\phi) \tilde{\phi}_1(v_{ij}) \right. \]

\[ - \cos(\theta) \sin(\phi) \tilde{\phi}_2(v_{ij}) + \sin(\theta) \tilde{\phi}_3(v_{ij}) \]

\[ = \pi i v \sum \beta_j \tilde{\phi}_j(v_{ij}) = \tilde{r}^{\beta \nu}(v_{ij}); \]

(17)

for the nonzero elements.

From Eq. (14), we see that by knowing which of the elements of the irrotational component are zero we can infer that the Fourier transform of the elements of the solenoidal component \( \tilde{S}_1, \tilde{S}_2, \) and \( \tilde{S}_3 \) are directly related to the Fourier transform of specific directional Radon projections. Therefore, the measurements \( r^{\theta \theta}, r^{\alpha \alpha}, \) and \( r^{\beta \beta} \) over a range of directions \( \theta \) covering a solid angle of \( 4\pi \) will specify the irrotational component of the tensor field. Also, from Eq. (14) we see that the measurements \( r^{\alpha \alpha}, r^{\beta \beta}, \) and \( r^{\beta \beta} \) over \( 4\pi \) will specify the solenoidal component.

2.4. Tensor field reconstruction using directional Radon projections

We first consider the straightforward reconstruction of elements of the tensor field from directional Radon projection measurements. The following sections will give formulæ for the reconstruction of solenoidal and irrotational components of the tensor field.

From Eq. (1), we can write the following matrix equation

\[ R(t; \theta) = \begin{bmatrix} r^{\theta \theta} & r^{\alpha \theta} & r^{\beta \theta} \\ r^{\alpha \theta} & r^{\alpha \alpha} & r^{\alpha \beta} \\ r^{\beta \theta} & r^{\beta \alpha} & r^{\beta \beta} \end{bmatrix} \]

(18)

where \( \tilde{r}_j(t; \theta) \) are the Radon projections of each individual element of the tensor field and

\[ \Theta = \begin{bmatrix} \sin \theta \cos \phi & -\sin \phi & -\cos \theta \cos \phi \\ \sin \theta \sin \phi & \cos \phi & -\cos \theta \sin \phi \\ \cos \theta & 0 & \sin \theta \end{bmatrix} \]

(19)

is an orthogonal matrix consisting of columns that correspond to the directional vectors \( \tilde{\theta}, \tilde{\alpha}, \) and \( \tilde{\beta} \), respectively.

Solving the matrix equation in Eq. (18) for \( \tilde{T} \), we have

\[ \tilde{T}(t; \theta) = \Theta R(t; \theta) \Theta^T. \]

(20)

From Eq. (20), the reconstruction of each element of \( \tilde{T}(t; \theta) \) can be written explicitly using Radon’s inversion formula [57]:

\[ \tilde{t}_j(x) = -\frac{1}{8\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e_i^r \tilde{r}_i^r(t; \theta) e_j^r \sin \theta \ d\theta \ d\phi, \]

(21)
where \( e^1 = \theta \), \( e^2 = \alpha \), and \( e^3 = \beta \). Using the notation \( r^{ij} \) in Eq. (21) to mean the same as \( r^{\alpha\alpha} \), the other elements \( r^{\alpha \tau} \) should be clear. The expression in Eq. (21) gives an equation for the reconstruction of each element of the tensor field in terms of the Radon inverse of a linear sum of directional Radon projection measurements.

2.5. Solenoidal reconstruction using directional Radon projections \( r^{\alpha \alpha}, r^{\alpha \beta}, \) and \( r^{\beta \beta} \)

Next we consider the reconstruction of the elements of the solenoidal component of the tensor field. From the projection theorem in Eq. (13) and the calculations for the elements in Eq. (14) we have the following result for the solenoidal component:

\[
\Theta^T \tilde{s}(v; \theta) = \begin{bmatrix}
0 & 0 & 0 \\
0 & r^{\alpha \alpha} & r^{\alpha \beta} \\
0 & r^{\alpha \beta} & r^{\beta \beta}
\end{bmatrix} (v; \theta).
\]  (22)

Solving for \( T^S \),

\[
T^S(v; \theta) = \Theta^T \tilde{s}(v; \theta).
\]  (23)

Therefore, the Fourier transform of each element of the solenoidal component of the tensor field is

\[
\tilde{f}_{lm}(v; \theta) = \sum_{k=1}^{3} e^{i \rho^{\alpha \alpha}(v; \theta) e_{lm}^\alpha}.
\]  (24)

Again using Radon’s inversion formula, we obtain the following filtered backprojection algorithm for each element of the solenoidal component of the tensor field

\[
t^S_{ij}(x) = -\frac{1}{8 \pi^2} \int_{4 \pi} \frac{\tilde{f}^{\alpha \alpha}(v; \theta)}{v} \sin \theta d \theta d \phi,
\]  (25)

\[
+ \frac{\partial}{\partial t} \left\{ \phi_i(t; \theta) \right\}_{t=\mu} \sin \theta d \theta d \phi.
\]  (26)

2.6. Irrotational reconstruction using directional Radon projections \( r^{\theta \theta}, r^{\theta \alpha}, \) and \( r^{\theta \beta} \)

The reconstruction of the irrotational component of the tensor field from directional Radon projections involves either reconstructing the elements of the vector potential function, from which the irrotational component of the tensor field can be determined from the gradient of the reconstructed potential, or directly reconstructing the irrotational component of the tensor field. Here we give expressions for the reconstruction of both the potential function and for the reconstruction of the elements of the irrotational component of the tensor field.

From the projection theorem we have the three equations, Eqs. (15–17), that give the Fourier transform of the directional Radon projections \( r^{\theta \theta}, r^{\theta \alpha}, \) and \( r^{\theta \beta} \) in terms of the Fourier transform of the three elements \( \tilde{\phi}_1, \tilde{\phi}_2, \) and \( \tilde{\phi}_3 \) of the vector potential. Writing this in matrix notation gives the following matrix equation

\[
\begin{bmatrix}
\tilde{p}^{\theta \theta} \\
\tilde{p}^{\theta \alpha} \\
\tilde{p}^{\theta \beta}
\end{bmatrix} (v; \theta) = \pi i \nu_t \Theta^T \begin{bmatrix}
\tilde{\phi}_1 \\
\tilde{\phi}_2 \\
\tilde{\phi}_3
\end{bmatrix} (v; \theta).
\]  (26)

Solving for \( \tilde{\phi}_1, \tilde{\phi}_2, \) and \( \tilde{\phi}_3 \) we have

\[
\tilde{\phi}_i(v; \theta) = \frac{1}{\pi i \nu_t} \left\{ \tilde{p}^{\theta \theta} (v; \theta) \right\}.
\]  (27)

Since \( 3^{-1} \{ 2 \pi i \nu_t \tilde{\phi}_i(v; \theta) \} = \tilde{\phi}_i(v; \theta) \), the expression in Eq. (27) leads to the following expression for the derivative of the projection of the elements of the vector potential

\[
\frac{\partial \tilde{\phi}_i}{\partial t} (v; \theta) = \theta \tilde{p}^{\theta \theta}(v; \theta) + 2 \tilde{\nu}_t \tilde{p}^{\theta \alpha}(v; \theta)
\]  (28)

The symbols \( \theta, \alpha, \) and \( \beta \) are elements of the directional vectors \( \theta, \alpha, \) and \( \beta \) respectively. Substituting this expression into the Radon inversion formula,

\[
\tilde{\phi}_i(x) = -\frac{1}{8 \pi^2} \int_{4 \pi} \frac{\tilde{f}^{\theta \theta}(v; \theta)}{v} \sin \theta d \theta d \phi,
\]  (29)

we have the following expression for the reconstruction of the elements of the vector potential

\[
\phi_i(x) = -\frac{1}{8 \pi^2} \int_{4 \pi} \frac{\tilde{f}^{\theta \theta}(v; \theta)}{v} \sin \theta d \theta d \phi.
\]  (30)

The reconstruction of the elements \( \phi_1, \phi_2, \) and \( \phi_3 \) of the vector potential of the irrotational component of the tensor field is given in terms of the backprojection of the derivative of a weighted sum of directional Radon measurements. The expression in Eq. (30) is a filtered backprojection-type algorithm in which the filter involves the implementation of a first derivative operation instead of the usual second derivative filtering operation of the Radon inversion formula.

Next we calculate the elements \( \tilde{f}^{\theta \beta}(x) = \frac{1}{2} (\partial \tilde{\phi}_i(x) + \partial \tilde{\phi}_j(x)) \) of the irrotational component by calculating the gradient of \( \tilde{\phi}_i \) in Eq. (30) noting that \( \partial h(x \cdot \theta)/\partial x_j = [\partial h(x \cdot \theta)/\partial t]_{t=\mu} \theta_j \). From the expression in Eq. (30) we can write the following filtered backprojection algorithm for the elements of the irrotational component of the tensor field.
Notice that the filtering operation for the reconstruction of the irrotational tensor elements involves a second derivative, whereas as we saw above the filtering operation for the reconstruction of the potential involves a first derivative.

3. Computer simulation

A computer simulation of a 3D reconstruction of a simulated cardiac diffusion tensor field was performed. This particular phantom was chosen in order to evaluate the potential for the application of MRI to the reconstruction of diffusion tensor fields in the myocardium. Of particular interest was to see if the direction of the principal vector associated with the maximum principal diffusion coefficient of the diffusion tensor could be used to specify the helical fiber structure of the myocardium.

3.1. Computer phantom

A circular cylindrical tube [9] was designed to simulate a diffusion tensor field [3] in a section of the mid-ventricular wall of the left ventricle. In the reference coordinate system the axis of the tube was aligned along the z-axis of the Cartesian coordinates \((x, y, z)\) with corresponding polar coordinates \((R, \theta, Z)\), where \(Z = z\). The inner radius of the tube was denoted as \(R_1\) and the outer radius of the tube was denoted as \(R_2\).

In our simulation we assumed that the fiber structure of the myocardium was helical [58–61]. Therefore, the computer model was designed so that the principal vectors of the diffusion tensor were referenced to a helical fiber structure with curvilinear material coordinates \((X_R, X_F, X_C)\), which are orthogonal in a stress-free state. The fiber axis \(X_F\) was located on the plane of the wall normal to the radial axis \(R\). The axis \(X_C\) was the cross-fiber in-plane axis and the axis \(X_R\) coincided with \(R\). The fiber angle \(\Psi(R)\) in the circumferential direction had a variation from endocardial to epicardial, which was continuous and linear:

\[
\Psi(R) = \left(\frac{R_2 - R}{R_2 - R_1}\right) \Psi_1 + \left(\frac{R - R_1}{R_2 - R_1}\right) \Psi_2
\]

where

\[
R = (x^2 + y^2)^{1/2}
\]

and \(\Psi_1\) and \(\Psi_2\) were the endocardial and epicardial fiber angles, respectively. Note that the function \(\Psi(R)\) was independent of the coordinate \(z\). In our simulations, \(\Psi_1 = +60^\circ\) and \(\Psi_2 = -60^\circ\) were the fiber axis angles at endocardial and epicardial walls, respectively [58–61].

It was assumed that the coordinate system \((X_R, X_F, X_C)\) rotates with the helical fibers relative to either the reference coordinate system \((x, y, z)\) or \((R, \theta, Z)\) and the rotation depends upon the transmural location in the short axis slice of the left ventricle, which in our simulation was a cylindrical tube. The coordinates of the vectors \(X_R, X_F, \) and \(X_C\) relative to the \((x, y, z)\) coordinate system are given by

\[
X_R = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 0\right),
\]

\[
X_F = \left(-\frac{y \cos \Psi}{\sqrt{x^2 + y^2}}, \frac{x \cos \Psi}{\sqrt{x^2 + y^2}}, \sin \Psi\right),
\]

and

\[
X_C = \left(\frac{y \sin \Psi}{\sqrt{x^2 + y^2}}, \frac{-x \sin \Psi}{\sqrt{x^2 + y^2}}, \cos \Psi\right).
\]

Our simulation used measured eigenvalues of a myocardial diffusion tensor, which were reported in [3] to be \(\lambda_1 = 0.7 \pm 0.1, \lambda_2 = 1.6 \pm 0.2, \) and \(\lambda_3 = 0.3 \pm 0.05 (10^{-3}\text{ mm}^2\text{ s}^{-1})\). It was assumed that the eigenvectors of the myocardial diffusion tensor lie along the fiber axes \(X_R, X_F, \) and \(X_C\), and the eigenvalues are the magnitude of the principal vectors such that \(\lambda_1\) was the magnitude of the principal vector along \(X_R\), \(\lambda_2\) was the magnitude of the principal vector along \(X_F\), and \(\lambda_3\) was the magnitude of the principal vector along \(X_C\).

The tensor field can be expressed in terms of its eigenvalue decomposition:

\[
D^{ij}(x) = \lambda_1 X_R^i X_R^j + \lambda_2 X_F^i X_F^j + \lambda_3 X_C^i X_C^j.
\]

Using this relationship gives the following expressions for the non-zero elements of the symmetric diffusion tensor field in the reference coordinate system \((x, y, z)\) within the cylindrical walls simulating the left ventricle:

\[
D(x, y, z) = \begin{bmatrix}
D^{11} & D^{12} & D^{13} \\
D^{21} & D^{22} & D^{23} \\
D^{31} & D^{32} & D^{33}
\end{bmatrix}
\]

where

\[
D^{11} = \frac{x^2 \lambda_1 + y^2 (\lambda_2 \cos^2 \Psi + \lambda_3 \sin^2 \Psi)}{x^2 + y^2},
\]

\[
D^{12} = \frac{xy (\lambda_1 - \lambda_2 \cos^2 \Psi - \lambda_3 \sin^2 \Psi)}{x^2 + y^2},
\]

\[
D^{13} = \frac{-y (\lambda_2 - \lambda_3) \sin 2\Psi}{2 \sqrt{x^2 + y^2}},
\]

\[
D^{22} = \frac{y^2 \lambda_1 + x^2 (\lambda_2 \cos^2 \Psi + \lambda_3 \sin^2 \Psi)}{x^2 + y^2},
\]

\[
D^{31} = \frac{y (\lambda_2 - \lambda_3) \cos 2\Psi}{2 \sqrt{x^2 + y^2}}.
\]
\[ D^{23} = D^{32} = \frac{x(\lambda_2 - \lambda_3) \sin 2\Psi}{2 \sqrt{x^2 + y^2}}, \text{ and} \]
\[ D^{33} = \lambda_2 \sin^2 \Psi + \lambda_3 \cos^2 \Psi. \]

It was assumed that the tensor field was zero for \( R < R_1, R > R_2, \) and the space beyond the length of the tube, which was all located outside the cylindrical wall.

For the simulations, the cylindrical phantom was placed at the center of a \( 32 \times 32 \times 32 \) array. The phantom had an inner radius \( R_1 \) equal to 8, an outer radius \( R_2 \) equal to 14, and a height equal to 8, such that slices 13–20 only contained non-zero data. This choice enabled avoidance of truncation problems during reconstruction. Also, simulations were performed using 16 \( \times \) 16 \( \times \) 16 and 64 \( \times \) 64 \( \times \) 64 arrays in order to evaluate partial volume effects due to voxel size.

### 3.2. Generating projections

To generate projection data a discrete to discrete calculation was used to form the directional planar projections of the simulated tensor field. Directional Radon planar projections were generated over half of a sphere for 16 \( \times \) 16 \( \times \) 16, 32 \( \times \) 32 \( \times \) 32, and 64 \( \times \) 64 \( \times \) 64 reconstructions. The voxel width was equal to the radial-sampling interval. The phantom array was first rotated to the correct angle, and then at each radial position the planar integral was formed by using linear interpolation and summing over all pixels in the plane that intersected the rotated phantom array. Projection data were uniformly sampled over a half-sphere: For plane that intersected the rotated phantom array. Projection using linear interpolation and summing over all pixels in the voxel width was equal to the radial-sampling interval. The voxel width was equal to the radial-sampling interval. The phantom was chosen to be the reconstructed tensor field for voxel \( i \) with corresponding eigenvector \( \mathbf{v}_j \), the following measures were calculated:

1) The average difference in the angle between a principal vector \( j = 1, 2, 3 \) of the tensor and the reconstructed diffusion tensor for \( N \) voxels was:

\[ \overline{\Delta \phi_j} = \sum_{i=1}^{N} \frac{\Delta \phi_{i,j}^{\text{phantom}, \text{recon}}}{N}, \]

where \( \Delta \phi_{i,j}^{\text{phantom}, \text{recon}} \) was the angle between the principal vector \( j \) in the reconstruction and the principal vector \( j \) in the phantom for voxel \( i \), that is, cos \( \Delta \phi_{i,j}^{\text{phantom}, \text{recon}} = \langle \mathbf{v}_j \rangle^{\text{phantom}} \cdot \langle \mathbf{v}_j \rangle^{\text{phantom}} \).

2) The average of the differences \( \Delta \lambda_j \) between the eigenvalue for the reconstruction and the eigenvalue for the reference phantom for the principal vector \( j \) was:

\[ \overline{\Delta \lambda_j} = \sum_{i=1}^{N} \frac{\Delta \lambda_{i,j}^{\text{phantom}, \text{recon}}}{N}, \]

where

\[ \Delta \lambda_{i,j}^{\text{phantom}, \text{recon}} = \left( \lambda^{\text{phantom}}_{i,j} - \lambda^{\text{phantom}}_{i,j} \right) \]

was the difference between the magnitude of the principal vector \( j \) for voxel \( i \) in the phantom and that of the reconstructed diffusion tensor normalized to the average magnitude for the \( N \) voxels in the reference phantom. In one comparison study the reference phantom was chosen to be the reconstructed tensor field. Note that \( \Delta \lambda_j \) is a bias, which can be negative.

3) The root-mean-square error in the angle differences for each principal vector \( j \) was
\[ \Delta \phi_j^{RMS} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (\Delta \phi_j)^2}. \]  
(48)

4) The root-mean-square error in the magnitude differences for each principal vector \( j \) was

\[ \Delta \lambda_j^{RMS} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (\Delta \lambda_j)^2}. \]  
(49)

The sums were performed over \( N \) pixels in a slice of the cylindrical wall of the tube simulating the left ventricular wall where the diffusion was non-zero. Also, error bars were calculated for each of the quantitative measures.

4. Results

4.1. Reconstruction of the tensor field

The reconstructions of the simulated cardiac tensor field for slice 18 of the \( 32 \times 32 \times 32 \) array are displayed on \( 32 \times 32 \) grids in Figs. 2–5. The first, second, and third eigenvalues of the tensor field are shown as a grayscale image in Fig. 2. The vector fields of the first, second, and third principal vectors are compared to those of the original phantom in Figs. 3, 4, and 5, respectively. Each vector is equal to the eigenvector times the eigenvalue in Fig. 2. When compared to the original phantom, the reconstruction results show some errors in the computer simulations due to the discrete implementation of a continuous analytical relationship, on which the filtered backprojection algorithm is based, and the interpolation inaccuracy in the rotation of the matrix array during the projection and backprojection operations.

A \( 16 \times 16 \times 16 \) simulation and a \( 64 \times 64 \times 64 \) simulation were also performed. A quantitative analysis of the \( 16 \times 16 \times 16 \) reconstruction is presented in Table 1. A quantitative analysis of the \( 32 \times 32 \times 32 \) reconstruction is presented in Table 2. A quantitative analysis of the \( 64 \times 64 \times 64 \) reconstruction is presented in Table 3. The \( 64 \times 64 \times 64 \) reconstruction comprises the greatest accuracy in all angle and magnitude measures of differences between the reconstruction and the original phantom. The increased errors in the \( 16 \times 16 \times 16 \) reconstruction and the \( 32 \times 32 \times 32 \) reconstruction are due to a partial volume effect caused by the larger pixels that are used in the \( 16 \times 16 \times 16 \) simulation and the \( 32 \times 32 \times 32 \) simulation. In general the errors were the smallest in the second principal vector. This is apparent in Figs. 3–5 where the second principal vector is totally radial, whereas the other principal vectors vary widely in direction by twisting and turning from voxel to voxel. Keep in mind that the principal vectors are ordered by magnitude so that the first principal vector should correspond to the \( X_F \) coordinate, the second should correspond to the \( X_K \) coordinate, and the third should correspond to the \( X_C \) coordinate.

4.2. Reconstruction of the solenoidal component

Figs. 6–8 present results of the reconstruction of the solenoidal component from directional Radon projections. The filtered backprojection algorithm in Eq. (25) was used to obtain the elements of the solenoidal component of the tensor field. The first, second, and third principal eigenvalues of the solenoidal component of the tensor field are shown as a grayscale image in Fig. 6. The vector field of first principal vectors for the reconstructed solenoidal component is presented in Fig. 7.

The first principal eigenvalues for the solenoidal component are much larger than for the irrotational component (see Fig. 9), thus the simulated diffusion tensor field has a first principal vector that is much more solenoidal than it is irrotational. Note that the second and third principal eigenvalues vectors can be negative. This seems most often to be the case in the region outside of the cylindrical phantom.

The first, second, and third rows of the solenoidal component of the reconstructed tensor field are displayed in Figs. 8a, 8b, and 8c, respectively. If we consider a constant vector \( \mathbf{v} \), then for any symmetric tensor \( t_{ij} \) the sum \( \sum_i \nu_i t_{ij} \) is a vector. If the symmetric tensor \( t_{ij} \) is divergence-free (solenoidal), that is \( \sum_j \partial t_{ij} / \partial x_j = 0, \forall i \), then \( \sum_i \nu_i t_{ij} \) is a vector that is also divergence free. The display in Fig. 8a is the vector field \( \sum_i \nu_i t_{ij} \) for \( \nu = (\nu_0, 0, 0) \), the display in Fig. 8b is a vector field for \( \nu = (0, \nu_0, 0) \) and the display in Fig. 8c is a vector field for \( \nu = (0, 0, \nu_0) \). Therefore, each display shows a divergence-free vector field that appears solenoidal without any sources or sinks.

Comparisons were made between the solenoidal reconstruction and the original phantom and the reconstructed tensor field. Only the first principal vector was compared. A quantitative analysis is presented in Table 4 for the comparison with the tensor field of the original phantom and a quantitative analysis is presented in Table 5 for the comparison with the reconstructed tensor field. In general, the angle and magnitude measures of differences were small. The differences between the solenoidal reconstruction and the reconstructed tensor field are smaller than the differences between the solenoidal reconstruction and the tensor field of the original phantom. It may be more reasonable to compare the reconstructed tensor field and its solenoidal reconstruction because both are reconstructions. The reason for this comparison was to determine if the first principal vector of the solenoidal reconstruction was sufficiently accurate to represent the myocardial fiber structure.

4.3. Reconstruction of the irrotational component

Figs. 9–11 present results of the reconstruction of the irrotational component from directional Radon projections. The filtered backprojection algorithms in Eq. (30) and Eq. (31) were used to reconstruct the vector potential and the irrotational component of the tensor field, respectively. Fig. 10
presents the reconstruction of the vector potential. The vector potential field has a large radial component and a small $z$-component. One would expect that the $z$-component should be zero. It is not, however, because of the finite length of the cylindrical phantom. The values of the first principal eigenvalue displayed in Fig. 9 are small compared to the values of the first principal eigenvalue for the solenoidal component in Fig. 6, whereas the second and third principal eigenvalues are comparable to the second and third principal eigenvalues for the solenoidal component.
Thus, the principal vector associated with the maximum eigenvalue of the solenoidal component is an indicator of fiber axis direction.

The first, second, and third rows of the irrotational component of the reconstructed tensor field are displayed in Figs. 11a, 11b, and 11c, respectively. As presented in Fig. 11, the “x” components \( t_{xx}^I, t_{xy}^I, t_{xz}^I \) and the “y” components \( t_{yx}^I, t_{yy}^I, t_{yz}^I \) of the irrotational component of the tensor field clearly have sources and sinks at the inner and outer boundaries of the cylindrical phantom in contrast to the solenoidal component. For the “z” components \( t_{zx}^I, t_{zy}^I, t_{zz}^I \), the vector field lies mostly along the z-axis. There are no sources in planes parallel to the xy-plane but there are sources and sinks at edges of the cylindrical phantom in the z-direction.

5. Discussion

Reconstruction formulae are presented for reconstructing solenoidal and irrotational components of 3D second order tensor fields from directional Radon projections. A simulation of the mid-ventricular region of the myocardium that showed that the principal vector corresponding to the maximum eigenvalue of the solenoidal component of the tensor field was in the direction of the simulated fiber bundle axis of the myocardium was presented. This information, or for that matter, the knowledge that the tensor field is totally solenoidal or totally irrotational, can be important a priori information since it can reduce the number of measurements required to reconstruct the tensor field. This work provides a framework and structure to tensor tomography experiments that can be useful for application of either analytical filter backprojection or iterative reconstruction algorithms in the processing of acquired data.
The approach taken in this study was to decompose the tensor field into a divergence-free component (solenoidal component) and a gradient of a vector potential (irrotational component) for which it is shown that such a decomposition is directly related to computed tomography. Fourier projection theorems were derived for the irrotational and solenoidal components, which provide relationships between these components and the directional Radon projection measurements. These relationships illustrate which directional measurements are totally related to the solenoidal component and which are totally related to the irrotational component. The decomposition is not necessary for obtaining a reconstruction algorithm for tensor fields, however, it provides a window into what tomography may be used for and what it may help to accomplish. Also, the relationships derived from the Fourier projection theorems lead to easy derivations of Fourier, filtered backprojection, and convolution backprojection reconstruction algorithms.

In the case of 2D vector tomography, Braun and Hauck [32] showed that the longitudinal and transverse measurements, which correspond to the use of directional measurements equal to $\theta$ and orthogonal to $\theta$, reconstruct the solenoidal and irrotational components of a vector field, respectively. In three dimensions, one might consider longitudinal measurements to be directional measurements equal to $u$ and transverse measurements to be directional measurements equal to two orthogonal vectors for which the choice is not unique. Prince [20] used the expression "irrotational measurements" to refer to the probe transform acquired using the probe $p = \theta$ (directional Radon transform $r^\theta$ of a vector field in our terminology). Similarly, he used the expression "solenoidal measurements" to refer to the pair of probe transform measurements acquired using probes $p_1 = \alpha$ and $p_2 = \beta$, which were both orthogonal to $\theta$. Here we see that for tensor fields the directional Radon projec-
tions, \( r^{\theta\theta}, r^{\theta\alpha}, \) and \( r^{\theta\beta} \), are **solenoidal measurements** and \( r^{\alpha\alpha}, r^{\alpha\beta}, \) and \( r^{\beta\beta} \) are **irrotational measurements**.

The tensor field decomposition can be especially useful in an MRI diffusion tensor experiment. If the diffusion tensor field is **totally solenoidal** or **totally irrotational** then three directional Radon projection measurements, \( r^{\theta\theta}, r^{\theta\alpha}, \) and \( r^{\theta\beta}, \) or \( r^{\alpha\alpha}, r^{\alpha\beta}, \) and \( r^{\beta\beta} \) can completely specify the tensor field, respectively. This is one half of the total number of measurements that is usually required to specify a diffusion tensor field. In the simulations the principal vector corresponding to the maximum eigenvalue of the solenoidal component was primarily aligned along the simulated fiber axis of the myocardium. Even though this can be very useful information, the diffusion tensor field was however not strictly solenoidal. The second and third principal eigenvalues of the solenoidal component were very close in magnitude to the second and third principal eigenvalues of the irrotational component of the simulated diffusion tensor field. If one considers a simple fiber model that is an incompressible bundle of fibers with preferential diffusion along the fibers, as in the simulation, then one can show that this is not solenoidal, even if \( \lambda_R = \lambda_C \).

**Table 1**  
Slice 9 of the 16 \( \times \) 16 \( \times \) 16 **tensor field** reconstruction. Reconstructions are compared to the original phantom for 124 voxels using the quantitative measures in Eqs. (45–49).

<table>
<thead>
<tr>
<th>Principal Vectors</th>
<th>First ( j = 1 )</th>
<th>Second ( j = 2 )</th>
<th>Third ( j = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \phi )</td>
<td>( 5.47^\circ \pm 4.15^\circ )</td>
<td>( 1.02^\circ \pm 0.80^\circ )</td>
<td>( 5.64^\circ \pm 4.12^\circ )</td>
</tr>
<tr>
<td>( \Delta \lambda )</td>
<td>( -0.24 \pm 0.10 )</td>
<td>( -0.15 \pm 0.13 )</td>
<td>( 0.20 \pm 0.26 )</td>
</tr>
<tr>
<td>( \Delta \phi_{\text{RMS}} )</td>
<td>( 6.89^\circ )</td>
<td>( 1.30^\circ )</td>
<td>( 7.00^\circ )</td>
</tr>
<tr>
<td>( \Delta \lambda_{\text{RMS}} \times 100 )</td>
<td>( 26% )</td>
<td>( 20% )</td>
<td>( 33% )</td>
</tr>
</tbody>
</table>
A purely solenoidal tensor field should satisfy: \( \text{div} \ T = 0 \). For the chosen cylindrical heart model this is equivalent to: \( \lambda_R = \lambda_F \cos^2 \Psi(R) + \lambda_C \sin^2 \Psi(R) \), where \( R \) is the radial coordinate and \( \Psi(R) \) in Eq. (32) is the fiber angle, which is a function of \( R \). The choice \( \lambda_R = \lambda_F = \lambda_C \) will satisfy \( \text{div} \ T = 0 \) for any \( R \). Another possibility is to assume that \( \Psi(R) \) is constant. However, both choices are not realistic for biological diffusion tensor fields. Some tensor elements in the \((X_R, X_F, X_C)\) system of coordinates can be assumed to be a function of \( R \). The most convenient choice is to let \( \lambda_R(R) \) be a function of only \( R \). In this case, \( \text{div} \ T = 0 \) is equivalent to \((R(d/dR) + 1)\lambda_R = \lambda_F \cos^2 \Psi(R) + \lambda_C \sin^2 \Psi(R) \). This is an ordinary differential equation that can be solved analytically in closed form for \( \lambda_R(R) \), if \( \Psi(R) \) is a known function of \( R \) and, \( \lambda_F \) and \( \lambda_C \) are known constant values.

The reconstruction of tensor fields is computationally more involved than the reconstruction of scalar fields. Even though the tensor field was symmetric in our simulations we reconstructed all of its nine elements to be certain. The reconstructions of the \(32 \times 32 \times 32\) tensor fields required approximately one hour on a Sun Ultra 60 (2 processors at 360 MHz). Half of the time involved the generation of simulated projections. The reconstruction of the nine elements took much more than nine times the time it took to perform one scalar reconstruction because of the additional multiplications of sines and cosines in the expression given in Eq. (21).

A cardiac diffusion tensor field was chosen for our simulations to investigate the potential application to MRI cardiac diffusion-tensor imaging. Our aim is to use MRI diffusion-tensor imaging to determine the fiber bundle orientation [2–5] in the myocardium from which one can specify a material axis for mechanical models [9–13] and identify conductive pathways [15] in order to develop electrical models of the heart. Thirty years ago Streeter and his colleagues quantified systematically the helical myocardial fiber structure [58–61] and more recent studies have been performed to confirm those results [62–64]. Comparison of

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**Table 2**
Slice 18 of the \(32 \times 32 \times 32\) tensor field reconstruction. Reconstructions are compared to the original phantom for 408 voxels using the quantitative measures in Eqs. (45–49).

<table>
<thead>
<tr>
<th>Principal Vectors</th>
<th>First</th>
<th>Second</th>
<th>Third</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D\phi )</td>
<td>2.77° ± 2.98°</td>
<td>0.55° ± 0.46°</td>
<td>2.89° ± 2.95°</td>
</tr>
<tr>
<td>( D\lambda )</td>
<td>−0.15 ± 0.11</td>
<td>−0.10 ± 0.12</td>
<td>0.13 ± 0.20</td>
</tr>
<tr>
<td>( \Delta\phi_{\text{RMS}} )</td>
<td>0.07°</td>
<td>0.27°</td>
<td>1.51°</td>
</tr>
<tr>
<td>( \Delta\lambda_{\text{RMS}} \times 100 )</td>
<td>18%</td>
<td>15%</td>
<td>23%</td>
</tr>
</tbody>
</table>

**Table 3**
Slice 36 of the \(64 \times 64 \times 64\) tensor field reconstruction. Reconstructions are compared to the original phantom for 1660 voxels using the quantitative measures in Eqs. (45–49).

<table>
<thead>
<tr>
<th>Principal Vectors</th>
<th>First</th>
<th>Second</th>
<th>Third</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D\phi )</td>
<td>0.82° ± 1.24°</td>
<td>0.22° ± 0.17°</td>
<td>0.90° ± 1.21°</td>
</tr>
<tr>
<td>( D\lambda )</td>
<td>−0.07 ± 0.1</td>
<td>−0.05 ± 0.1</td>
<td>0.03 ± 0.12</td>
</tr>
<tr>
<td>( \Delta\phi_{\text{RMS}} )</td>
<td>1.49°</td>
<td>0.17°</td>
<td>1.51°</td>
</tr>
<tr>
<td>( \Delta\lambda_{\text{RMS}} \times 100 )</td>
<td>12%</td>
<td>11%</td>
<td>12%</td>
</tr>
</tbody>
</table>

---

First Principal Eigenvalue

Second Principal Eigenvalue

Third Principal Eigenvalue

Fig. 6. Solenoidal reconstruction from directional radon projections. Eigenvalues of the solenoidal component of the tensor field are presented as a grayscale display.
diffusion-tensor MRI and histologic studies have shown that the first principal eigenvector of the diffusion tensor is nearly parallel to the fiber orientation in myocardial tissue [65,66].

The model of the heart muscle mechanics is based on the passive and active behavior of skeletal muscle [9–13] in which the muscle is described by a quasi-incompressible transversely isotropic hyperelastic material [9,67,68]. The transversely isotropic material is defined relative to fiber bundle sheaths of a specified helical orientation in a manner similar to those used for skeletal muscle [67,68]. Not only can MRI provide this structural information about fiber orientation (from diffusion tensor imaging) needed for the development of mechanical models, but it can also provide in vivo strain measurements [6–8] that can be compared to strain calculations obtained from the model. The characterization of cardiac deformation through strain measurements is an important part of the process of determining cardiac viability [69], quantifying ischemic injury, and evaluating perfusion by correlating perfusion with measures of strain [70].

The specification of cardiac diffusion tensor fields may also have potential application in the reconstruction of conductivity tensor fields in biological tissue. It is believed that there is strong correlation between the principal vectors of the diffusion tensor and conductivity tensor in the myocardium [15]. This would be useful in the specification of
the forward operation in solving the inverse magnetocardiography (MCG) problem [71].

Tensor tomography could also have important application in brain imaging. Diffusion-tensor MRI is already becoming an important application in the diagnosis of several brain disorders. The primary application is the diagnosis of acute stroke and ischemia [72–75]. It has also been found to be useful in diagnosing several other disorders. One of the most recent developments is the mapping of axon tracts in white matter to better characterize white matter disease and to determine the correlation between function and morphology [76,77]. Diffusion-tensor MRI has also been used to determine the correlation between activation and white matter connectivity [78,79]. Additional applications have been developed for surgical planning, and the study of remodeling of function following brain injury [80].

Therefore, extensive work has already been accomplished developing techniques for measuring diffusion tensor fields using MRI without computed tomography. The question is, will tensor computed tomography provide a more accurate and more efficient method for obtaining distributions of diffusion tensors in tissue, using MRI? If it is known a priori that the tensor field is totally solenoidal or totally irrotational then we know that the field can be specified in half the number of measurements. If it is not known to be one or the other then a complete set of projection measurements equal to the number of conventional mea-
measurements is needed to obtain the reconstructed tensor field. It may be under certain conditions that this is still more efficient. However, comparisons with established MRI techniques as well as with new approaches being pursued must be made. In one new approach [16] projections are formed of diffusion weighted MRI images (not projections of the tensor field as we are proposing here). These projections are reconstructed to form diffusion-weighted images. With the application of different diffusion gradient weights the spatial distribution of the diffusion tensor can be calculated.

To form projections of the diffusion tensor field using MRI, a linear approximation of the exponential attenuator in the expression for the diffusion-weighted signal must be made [19]. Also, manipulations of measurements must be made in order to obtain desired scalar projection measurements of the diffusion tensor field. The reconstruction of the projections yields a spin density weighted diffusion tensor field. It is necessary to divide this by the reconstruction of the spin density to obtain the actual diffusion tensor field. Work still must be performed to verify whether this less direct approach is sufficiently accurate when compared with conventional spin warp imaging or new MRI tomographic techniques. We are also investigating the use of iterative reconstruction algorithms that have the capacity to reconstruct the diffusion tensor field without any approximations from the nonlinear projection measurements [56]. This approach may provide a more accurate result than the one obtained using an approximation to form the projections for the filtered backprojection method presented in this paper.

The results presented here may have applications in other areas of medical imaging as well. For example, tensor tomography may be useful for in vivo mapping of cardiac deformation since it may provide a technique for obtaining three-dimensional distributions of strain and stress tensor fields in the myocardium. It is possible that MRI tensor tomography is more efficient and less sensitive to strain modulation than methods previously proposed for imaging of diffusion tensor fields in the heart [2–5].

Table 4
Slice 36 of the 64 × 64 × 64 solenoidal reconstruction. Reconstructions are compared to the original phantom using the quantitative measures in Eqs. (45–49).

<table>
<thead>
<tr>
<th>Principal Vectors</th>
<th>First</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \phi_j$</td>
<td>$5.62^\circ \pm 2.85^\circ$</td>
</tr>
<tr>
<td>$\Delta \lambda_j$</td>
<td>$-0.19 \pm 0.12$</td>
</tr>
<tr>
<td>$\Delta \phi_j$</td>
<td>$6.30^\circ$</td>
</tr>
<tr>
<td>$\Delta \lambda_j$</td>
<td>$22%$</td>
</tr>
</tbody>
</table>

Table 5
Slice 36 of the 64 × 64 × 64 solenoidal reconstruction. Reconstructions are compared to the reconstructed tensor field results using the quantitative measures in Eqs. (45–49).

<table>
<thead>
<tr>
<th>Principal Vectors</th>
<th>First</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \phi_j$</td>
<td>$5.15^\circ \pm 2.40^\circ$</td>
</tr>
<tr>
<td>$\Delta \lambda_j$</td>
<td>$-0.13 \pm 0.04$</td>
</tr>
<tr>
<td>$\Delta \phi_j$</td>
<td>$5.67^\circ$</td>
</tr>
<tr>
<td>$\Delta \lambda_j$</td>
<td>$13%$</td>
</tr>
</tbody>
</table>

Fig. 9. Irrotational reconstruction from directional radon projections. Eigenvalues of the irrotational component of the tensor field are presented as a grayscale display.
interesting to note that the application of various non-tomographic techniques using MRI have already resulted in advancements towards obtaining three-dimensional strain maps of the myocardium [81–85]. Presently, one question must be answered before tensor tomography can be applied to this problem: Can projections of strain and stress tensor fields be measured with a particular imaging modality?

In summary, tensor tomography builds upon the significant amount of work performed over the last ten years in vector tomography. The continuation of work in this area offers the potential for significant new mathematical developments in the field of inverse problems. The development of new algorithms in tensor tomography may have important application to in vivo mapping of brain diffusion tensors and cardiac strain, stress, diffusion, and conductivity tensors using MRI. Also, there is the potential for application to other imaging modalities that use acoustic or electrical magnetic radiation to measure tensor quantities in biological tissues. Information about the diffusion tensor field in the myocardium has direct application to both modeling of the mechanical properties, which are defined relative to the fiber bundle position and orientation in the heart which can be determined from a map of the diffusion tensor field, and to electrical conductivity properties in the heart which can also be inferred from the diffusion tensor field. These aspects of tensor tomography present fascinating areas of potential research. Most certainly the advent of tensor

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**Fig. 10.** Irrotational reconstruction from directional Radon projections. The vector field of the reconstruction of one slice of the vector potential from directional Radon projections. The isometric view is shown on the top and the transverse view is shown on the bottom.

**Fig. 11a.** Irrotational reconstruction from directional Radon projections. Vector field consisting of ($t_{xx}$, $t_{xy}$, $t_{xz}$). The isometric view is shown on the top and the transaxial view is shown on the bottom.
tomography has created a rich arena for mathematical development.

Appendix

This appendix gives details of how the MRI data are processed in order to organize the MRI data into a projection reconstruction format. The tensor tomography approach involves a linearization of the signal equation in order to formulate the measurements into projections of a 3D tensor field. Also, the formation of cross terms $r_{ua}^h(\xi;\theta)$ cannot be obtained from a single application of diffusion gradients but must be obtained from a combination of measurements. This was first presented in [19] for the two-dimensional case. Here it is presented for 3D planar Radon projections.

A.1. The linear approximation

For a readout gradient $G_r$, the signal during readout is

$$s_{\omega}(t) = \int \rho(\chi)e^{i\chi X G_r}e^{-b\omega^TD(x)\omega}dx,$$  \hspace{1cm} (A1)

where $\rho$ is the spin density, $\gamma$ is the gyromagnetic ratio, $\omega$ is the direction of the application of the diffusion weighting gradient, $D$ is the diffusion tensor, and

$$b = \gamma^2G^2\Lambda^2(\Delta - \Lambda/3),$$  \hspace{1cm} (A2)

where $G$ is the amplitude of the diffusion weighting gradient, $\Lambda$ is the length of one lobe of the diffusion pulse, and $\Delta$ is the separation between the start of the two gradient pulses. Note that initially we here use the notation $\omega^TD(x)\omega$. 

Fig. 11b. Irrotational reconstruction from directional radon projections. Vector field consisting of $(t_{yx}, t_{zy}, t_{yz})$. The isometric view is shown on the top and the transaxial view is shown on the bottom.

Fig. 11c. Irrotational reconstruction from directional radon projections. Vector field consisting of $(t_{zx}, t_{zy}, t_{zz})$. The isometric view is shown on the top and the transaxial view is shown on the bottom.
to be the same as a \( \sum_{i,j} \omega_i D^{ij}(x) \omega_j \), which is the notation we used throughout the paper for the contraction of a tensor with two directional vectors.

Taking the Fourier transform of the signal: \( S_\omega(f) = \mathcal{F}\{s_\omega(t)\} \), we have

\[
S_\omega(f) = \int \rho(x) e^{-b x^T D(x) \omega} e^{-2\pi i f \cdot x} \, dx
\]

\[
- f \, dx.
\]

(A3)

Finally, we define \( g_\omega(\xi; \theta) = (\gamma/G_i)/(2\pi)) S_\omega(\gamma\gamma_i)/ (2\pi)) \), which gives the expression

\[
g_\omega(\xi; \theta) = \int \rho(x) e^{-b x^T D(x) \omega} \delta(x \cdot \theta - \xi) \, dx.
\]

(A4)

If the argument of the exponential in (A6) is small, then the exponential can be approximated by

\[
e^{-b x^T D(x) \omega} \approx 1 - b \omega^T D(x) \omega,
\]

(A7)

which yields an approximation to \( g_\omega \):

\[
g_\omega(\xi; \theta) = \int \rho(x) \delta(x \cdot \theta - \xi) \, dx
\]

\[
- \int \rho(x) b \omega^T D(x) \omega \delta(x \cdot \theta - \xi) \, dx
\]

\[
= a_\omega(\xi; \theta)
\]

(A8)

where \( a_\omega \) is the approximation to \( g_\omega \). Now consider two measurements \( g_0(\xi; \theta) \) acquired with \( G = 0 \), and \( g_\omega(\xi; \theta) \) with \( G \neq 0 \) applied in the direction of the unit vector \( \omega \). Then

\[
[g_0(\xi; \theta) - g_\omega(\xi; \theta)] b = [g_0(\xi; \theta)
\]

\[
- a_\omega(\xi; \theta)] b
\]

(A9)

\[
= \int \rho(x) \delta(x \cdot \theta - \xi) \, dx
\]

(A10)

\[
= \int \rho(x) \left[ \sum_{ij} \omega_i D^{ij}(x) \omega_j \right] \delta(x \cdot \theta - \xi) \, dx
\]

\[
= p^{\omega 0}(\xi; \theta)
\]

(A11)

is precisely the approximated Radon projection of the product of the spin density times the contraction of the tensor \( D \) with the unit vector \( \omega \).

Note that the expression in Eq. (A11) is not quite the Radon projection \( r^{\omega 0}(\xi; \theta) \) of the tensor field as defined in Eq. (1) because of the weighting by the spin density. Thus we give here the notation \( p^{\omega 0}(\xi; \theta) \) for the spin density weighted tensor field Radon projection. By first reconstructing according to the algorithms presented in this paper the spin density weighted projections of the tensor field from the appropriate combination of the approximated projections \( [g_0(\xi; \theta) - g_\omega(\xi; \theta)] b \) then dividing that reconstruction by the reconstruction of the spin density \( \rho \) yields the reconstruction of the tensor field itself.

A.2. The method of obtaining directional measurements

MRI measurements prescribed in Eq. (A11) can only provide the diagonal elements \( p^{\omega 0}(\xi; \theta) \) of the directional projection data. However, for the algorithms presented in this paper, it is also necessary to obtain the non-diagonal elements, such as \( p^{\omega \alpha}(\xi; \theta) \). Therefore, it is necessary to somehow obtain the non-diagonal elements from a combination of diagonal MRI projection measurements. Here we make the derivation for one example.

Let \( \eta = (\alpha + \beta)/\sqrt{2} \). With the substitution \( \omega = \eta \) into Eq. (A11), it is easy to show that

\[
p^{\eta \eta}(\xi; \theta) = [p^{00}(\xi; \theta) + 2p^{\alpha \beta}(\xi; \theta)]
\]

\[
+ p^{\beta \beta}(\xi; \theta)/2.
\]

Hence,

\[
p^{\alpha \beta}(\xi; \theta) = [2p^{\eta \eta}(\xi; \theta) - p^{\alpha \alpha}(\xi; \theta)
\]

\[
- p^{\beta \beta}(\xi; \theta)]/2.
\]

(A13)

Other components such as \( p^{\alpha \alpha}(\xi; \theta) \) and \( p^{\beta \beta}(\xi; \theta) \) are obtained in a similar way using vectors \( \gamma = (\theta + \omega)/\sqrt{2} \) and \( \psi = (\theta - \omega)/\sqrt{2} \), respectively.

Consequently, to form the projections the diffusion gradients must be prescribed and the following projection measurements need to be made for each projection angle \( \theta = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). First, without any diffusion gradient, we obtain the following projections

\[
g_0(\xi; \theta) = \int \rho(x) \delta(x \cdot \theta - \xi) \, dx.
\]

(A14)

Then, with diffusion gradients \( \partial G = G(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), we obtain the following projections

\[
p^{\alpha \beta}(\xi; \theta) \approx [g_0(\xi; \theta) - g_\omega(\xi; \theta)] b,
\]

(A15)

where \( p^{\alpha \beta}(\xi; \theta) \) is modeled to be
\[ p^{0\alpha}(\xi; \theta) = \int \rho(x) \left[ \sum_{i,j} \theta_i D^{ij}(x) \theta_j \right] \delta(x \cdot \theta - \xi) \, dx. \] (A16)

With the diffusion gradients \( G_D = G(-\sin \phi, \cos \phi, 0) \), we obtain
\[ p^{\alpha\alpha}(\xi; \theta) = \frac{[g_{0\alpha}(\xi; \theta) - g_{0\alpha}(\xi; \theta)]/b, \] (A17)
where \( p^{\alpha\alpha}(\xi; \theta) \) is modeled to be
\[ p^{\alpha\alpha}(\xi; \theta) = \int \rho(x) \left[ \sum_{i,j} \alpha_i D^{ij}(x) \alpha_j \right] \delta(x \cdot \theta - \xi) \, dx. \] (A18)

Likewise with the application of the diffusion gradients \( G_D = G(-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta) \) we obtain measurements for \( p^{\beta\beta}(\xi; \theta) \)
\[ p^{\beta\beta}(\xi; \theta) = \frac{[g_{0\beta}(\xi; \theta) - g_{0\beta}(\xi; \theta)]/b}. \] (A19)

Next we need the off diagonal directional measurements. With application of the diffusion gradients \( G_D = G(\sin \theta \cos \phi - \sin \phi \cos \theta \sin \phi, \sin \theta \sin \phi + \cos \phi \cos \theta \sin \phi, \cos \theta + \sin \theta) \) we obtain measurements for \( p^{\gamma\gamma}(\xi; \theta) \), with \( G_D = G(\sin \theta \cos \phi - \cos \phi \sin \theta \cos \phi, \sin \theta \sin \phi - \cos \phi \cos \theta \sin \phi, \cos \theta - \sin \theta) \) we obtain \( p^{\alpha\alpha}(\xi; \theta) \), and with \( G_D = G(-\sin \phi - \cos \phi \cos \theta \cos \phi, \sin \theta \sin \phi - \cos \phi \cos \theta \sin \phi, \cos \theta + \sin \theta) \) we obtain \( p^{\rho\rho}(\xi; \theta) \). In all of these measurements, the read out gradient is \( G_r = G(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). To form the projections \( p^{0\alpha}(\xi; \theta) \) we must subtract from \( p^{\gamma\gamma}(\xi; \theta) \) the projections \( p^{0\alpha}(\xi; \theta) \) and \( p^{\alpha\alpha}(\xi; \theta) \) then divide the result by two:
\[ p^{0\alpha}(\xi; \theta) = \frac{[2p^{0\alpha}(\xi; \theta) - p^{0\alpha}(\xi; \theta)]/2} 2. \] (A20)

To form the projections \( p^{0\beta}(\xi; \theta) \):
\[ p^{0\beta}(\xi; \theta) = \frac{[2p^{0\beta}(\xi; \theta) - p^{0\beta}(\xi; \theta)]/2} 2. \] (A21)

To form the projections \( p^{\alpha\beta}(\xi; \theta) \):
\[ p^{\alpha\beta}(\xi; \theta) = \frac{[2p^{\alpha\beta}(\xi; \theta) - p^{\alpha\beta}(\xi; \theta)]/2} 2. \] (A22)

After reconstructing using the projection measurements \( p^{0\alpha}(\xi; \theta), p^{\alpha\alpha}(\xi; \theta), p^{\beta\beta}(\xi; \theta), p^{\alpha\beta}(\xi; \theta), \) and \( p^{\alpha\beta}(\xi; \theta) \) the results must be divided by the reconstruction of the spin density \( \rho(x) \) to obtain the reconstructed tensor distribution.

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