Analytical Reconstruction Formula for One-Dimensional Compton Camera

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Abstract
The Compton camera has been proposed as an alternative to the Anger camera in SPECT. The advantage of the Compton camera is its high geometric efficiency due to electronic collimation. The Compton camera collects projections that are integrals over cone surfaces. Although some progress has been made toward image reconstruction from cone projections, at present no filtered backprojection algorithm exists. This paper investigates a simple 2D version of the imaging problem. An analytical formula is developed for 2D reconstruction from data acquired by a 1D Compton camera that consists of two linear detectors, one behind the other. Coincidence photon detection allows the localization of the 2D source distribution to two lines in the shape of a "V" with the vertex on the front detector. A set of "V" projection data can be divided into subsets whose elements can be viewed as line-integrals of the original image added with its mirrored shear transformation. If the detector has infinite extent, reconstruction of the original image is possible using data from only one such subset.

I. INTRODUCTION

Conventional gamma cameras used in SPECT localize gamma emitters by a mechanical collimator. This technique leads to very low efficiency because only a fraction of the radiation is recorded through the collimator. Also at any given time only one view of the object is obtained, so the camera needs to move relative to the patient in order to collect all the data necessary for image reconstruction.

A new type of gamma camera for SPECT, originally proposed by Everett et al. [1] and by Singh [2] and further investigated in [6-8], utilizes Compton scattering for gamma source localization. Using electronic collimation as an alternative to mechanical collimation provides both high efficiency and multiple projections of the object. The camera consists of two plane gamma detectors positioned one behind the other. An incident photon undergoes Compton scattering in detector 1 and is absorbed by detector 2 (Fig. 1). Corresponding positions \( x \) and \( x' \) as well as energy \( \Delta E \) deposited in the first detector are measured. Angle \( \beta \) can be found using

\[
\cos \beta = 1 - \frac{mc^2 \Delta E}{(E - \Delta E)E}.
\]

Assuming that \( x \), \( x' \) and \( \beta \) are known, one can conclude that the gamma source is located somewhere on the cone surface (Fig. 2). The relationship between 3D gamma source distribution \( f(x,z) \) and the rate of photon counting \( q(x, x', \beta) \) for specific values of \( x \), \( x' \), \( \beta \) is given by

\[
q(x, x', \beta) \propto \int_{cone} f dA. \tag{2}
\]

Data acquired by the camera can be considered as samples of \( q(x, x', \beta) \) and are usually called cone projections.

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Although a few different approaches to image reconstruction from cone projections have been proposed [3, 4], all of them rely on iterative techniques, and are very time consuming. On the other hand, the complicated geometry associated with cone projections is a major obstacle to finding analytical algorithms, such as filtered backprojection (FBP). Clearly, new approaches must be found for a successful solution of the reconstruction problem.

In this paper we investigate a simple 2D imaging problem. We first describe a one-dimensional Compton camera consisting of two linear detectors, which allows the localization of the 2D source distribution to two lines in the shape of a “V” with the vertex on the front detector. We then show how a set of data (called “V”-projections) can be divided into subsets in such a way that the image can be reconstructed using “V”-projections from only one such subset. Finally, we present the results of computer simulations.

II. THEORY

A. One-dimensional Compton camera

A one-dimensional Compton camera consists of two linear gamma detectors, one behind the other (Fig. 3). An incident photon undergoes Compton scattering in the first detector, and is absorbed by the second detector. The corresponding positions \( x, x' \) and scattering angle \( \beta \) are measured. The camera is capable of counting photons for the different values of \( x, x' \) and \( \beta \). Therefore, projection data can be represented by function \( q(x, x', \beta) \). If \( x, x' \) and \( \beta \) are known, the source location of the collection of photons for this projection is limited by two semilines, \( L_1 \) and \( L_2 \), with a common vertex on the front detector (Fig. 4).

The relationship between the gamma source distribution \( f(x, z) \) and the projection data \( q(x, x', \beta) \) is given by the following:

\[
q(x, x', \beta) = |\cos \theta_1| \int_{L_1} f(x, z) \, dl + |\cos \theta_2| \int_{L_2} f(x, z) \, dl ,
\]

(3)

where \( \theta_1 \) and \( \theta_2 \) are the angles between \( L_1 \) and \( L_2 \), respectively, and a normal to the detector (see Fig.3).

Although the variables \( x, x' \) and \( \beta \) work well to label individual “V”-projections, they are not convenient to describe the relationship between the gamma source distribution and the projection data. Therefore, we introduce a new set of variables, \( x, k' = \tan \theta_1 \) and \( k'' = \tan \theta_2 \) (Fig. 5). The relationship between the “V”-projections and the gamma source distribution (3) can now be written as

\[
q(x, k', k'') = \int_{0}^{\infty} f(x + k' z, z) \, dz + \int_{0}^{\infty} f(x + k'' z, z) \, dz .
\]

(4)

We will also use \( K = k' + k'' \) and \( k = k' \), so that \( q(x, k', k'') = q(x, k, K - k) \).

B. Relationship between “V”-projections and line projections.

The reconstruction approach is based on the fact that the “V”-projection \( q(x, k, K - k) \) of function \( f(x, z) (z \geq 0) \) is equal to the line projection

\[
p_K(x, k) = \int_{-\infty}^{\infty} f_K(x + k z, z) \, dz
\]

of the function \( f_K(x, z) \), defined as

\[
f_K(x, z) = \begin{cases} f(x, z) & \text{if } z \geq 0 \\ f(x - K z, -z) & \text{if } z < 0 \end{cases}.
\]

(5)
To demonstrate this, consider the second integral in (4) when \( k'' = K - k \). Changing variable of integration from \( z \) to \( -z \) and using the definition of \( f_K(x, z) \), we have

\[
\int_0^{\infty} f(x + (K - k)z, z) dz = \int_0^{\infty} f(x - Kz + k, -z) dz
\]

\[
= \int_0^{\infty} f_K(x + k, z) dz.
\]

Substituting this back into (4) gives

\[
q(x, k, K - k) = \int_0^{\infty} (f(x + k, z) + f(x + (K - k)z, z)) dz
\]

\[
= \int_0^{\infty} f_K(x + k, z) dz + \int_0^{\infty} f_K(x + k, z) dz
\]

\[
= p_K(x, k).
\]

This result has a simple geometrical interpretation. Let us consider first the case with \( K=0 \). As follows from its definition, \( f_0(x, z) \) is symmetric with respect to the \( x \) axis (\( f_0(x, z) = f_0(x, -z) \)). An example of such a function is shown on Fig. 6(a). AOB represents one "V"-projection \( q(x, k, -k) \). Since lines \( AA' \) and \( BB' \) have opposite slope (\( 1/k \) and \(-1/k\)), they are also symmetric with respect to the \( x \) axis, and line-projections \( AO, BO \) are equal to line-projections \( OB', OA' \) correspondingly. Therefore, line-projections \( AA' \) and \( BB' \) of function \( f_0(x, z) \) are equal to "V"-projection AOB of function \( f(x, z) \).

It is more difficult to visualize the case with \( K \neq 0 \) because \( f_K(x, z) \) in the lower half represents a mirrored shear transformation of \( f(x, z) \). An example with \( K=-2 \) is presented in Fig. 6(b). As in the previous case, line-projections \( AA' \) and \( BB' \) of function \( f_{-2}(x, z) \) are equal to "V"-projection AOB of function \( f(x, z) \).

![Figure 5. Two different sets of variables used to label individual "V"-projections: (a) \( x, x', \beta \); (b) \( x, k', k'' \).](image)

![Figure 6. Geometrical interpretation of the relationship between "V"-projections of function \( f(x, z) \) and line-projections of function \( f_K(x, z) \): (a) \( K=0 \); (b) \( K=-2 \).](image)
C. Reconstruction algorithm

Let us consider a subset of "V"-projections
\[ S_K = \{ q(x, k', k'') | K = k' + k'' \}, \] (6)
for some value of \( K \). As it was proved before every "V"-projection \( q(x, k, K - k) \) contained in this subset is equal to the line projection \( p_K(x, k) \) of a modified function \( f_K(x, z) \). In other words, \( S_K \) can be considered as a set of all line projections of the function \( f_K(x, z) \). This fact allows us to use any conventional algorithm (for instance, FBP) to reconstruct \( f_K(x, z) \) from data contained in \( S_K \). Since \( f_K(x, z) \) has the same values as \( f(x, z) \) for positive \( z \), we conclude that a gamma source distribution can be reconstructed from a subset \( S_K \) of "V"-projections for any \( K \) using the FBP algorithm.

III. COMPUTER SIMULATION

In computer simulations we used the phantom \( f(x, z) \) shown in Fig. 7(a). "V"-projections were generated exactly with analytical formulae. For a fixed value of \( K \), "V"-projections \( q(x, k, K - k) = p_K(x, k) \) from the corresponding subset \( S_K \) were used to estimate X-ray projections
\[ \hat{p}_\theta(t) = \int \int f_K(x, z) \delta(x \sin \theta + z \cos \theta - t) dx \, dz \] (7)
as follows:
\[ \hat{p}_\theta(t) = \frac{1}{|\cos \theta|} p_K(t / \cos \theta, \tan \theta), \quad \pi / 2 < \theta < \pi / 2. \] (8)
Finally, the standard FBP algorithm was used to reconstruct \( f_K(x, z) \) from \( \hat{p}_\theta(t) \).

Figure 7(b) shows the image reconstructed from the subset \( S_{1/2} \) of "V"-projection data. In accordance with the theory, the original phantom is reconstructed in the upper half while its transformed copy is obtained in the lower half.

IV. CONCLUSION

The Compton camera was proposed for 3D medical imaging by Everett et al. [1] and by Singh [2]. Utilizing electronic collimation instead of mechanical collimation in conventional cameras leads to better efficiency and allows one to obtain multiple views of the object without moving the detector. The Compton camera collects projections that are integrals over cone surfaces through the object. Although some progress has been made toward image reconstruction from cone projections [3-5], all approaches based on iterative techniques were found to be very time consuming. New algorithms must be found for efficient reconstruction.

As a first step in that direction, a simple one-dimensional Compton camera has been investigated, which consists of two linear gamma detectors, one behind the other. Coincidence photon detection allows the localization of the 2D source distribution to two lines in the shape of a "V" with the vertex on the front detector. It was shown that a set of "V" projection data can be divided into subsets whose elements can be viewed as line-integrals of the original image added to its mirrored shear transformation. Applying the conventional FBP algorithm to data from one such subset reconstructs the original image in the upper half plane. The sheared transformed copy of the image is reconstructed in the lower half, and is therefore separated from the true copy.

To apply this approach in practice one would need to build a device capable of measuring "V"-projections. One way to do that is to use a conventional Compton camera in combination with a slit collimator which would reduce cone...
uncertainty to two lines. Although mechanical collimation reduces efficiency, such a camera would still have advantage of obtaining multiple views of the object.

Energy resolution of the gamma detector as well as noise and scatter will affect the reconstruction. It is unlikely that image of good quality could be obtained from just one subset $S_X$ of projection data. Simple addition of images reconstructed from different subsets would improve statistics in practical imaging. Also some modification of the algorithm will be required due to practical limits on detector size.

APPENDIX

The concept of “V”-projections as well as the reconstruction technique described in the paper can be extended to 3D as follows. Consider three-dimensional space with $(x_1, x_2, z)$ coordinates and gamma source distribution described by a function $f(x, z) (z \geq 0)$ (we use bold notation to represent 2D vectors and points lying in $x_1, x_2$ plane). An arbitrary “V”-projection consisting of two semilines $L', L''$ with common vertex on $z=0$ plane can be specified by a point $x$ and two vectors $k'$ and $k''$ (Fig. 8).

The relationship between the gamma source $f(x,z)$ and projection data $q(x, k', k'')$ can be obtained from (4) simply by changing scalars $x, k'$ and $k''$ to vectors $x, k'$ and $k''$:

$$q(x, k', k'') = \int_0^\infty \int_0^\infty f(x + k'z, z)dz + \int_0^\infty \int_0^\infty f(x + k''z, z)dz.$$  

Using this fact one can obtain the following relationship between the 2D Fourier transform of $q(x, k, K-k)$ with respect to $x$ and the 3D Fourier transform of $f_K(x, z)$:

$$Q(\mathbf{f}, k, K-k) = F_K(\mathbf{f}, -\mathbf{k})$$  

This shows that $f_K(x, z)$ (and, therefore, $f(x,z)$) can be reconstructed from “V”-projections $q(x, k, K-k)$ measured for a sufficiently large set of $k$ values.

REFERENCES


