IMAGE RECONSTRUCTION USING A GENERALIZED NATURAL PIXEL BASIS

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Abstract

The solution \( q \) of the imaging equation \( Mq = FGq = p \) (\( F \) is the projector and \( G \) is a generalized backprojector) is determined using least squares, thus various basis functions can be used as an expansion for the reconstructed image. In this paper a generalized natural pixel basis is chosen to allow flexibility in formulating the vector space for the solution \( q \). The singular value decomposition (SVD) method is used to solve for \( q \), and the final image is obtained by backprojecting \( q \): \( \rho = Gq \), and sampling \( \rho \) at a discrete array of points. Truncated parallel and non-truncated fan beam projection measurements were used to demonstrate that the solution \( q \) to \( Mq = FGq = p \) can be defined wherein, for example, if \( F \) is a fan beam projection operator, \( G \) can be a parallel backprojection operator defined based upon natural pixels. It is demonstrated that different backprojection geometries can give almost equivalent reconstructions of non-truncated projections. For truncated projections the estimation of \( q \) that covers the entire projection of the object is effective in reducing ring artifacts; however, using more projection bins is much more effective in preserving the resolution than is increasing the projection bin width. Also, a generalized natural pixel basis better models the geometric response of a collimator used in SPECT, therefore reconstructions of fan beam projections using generalized natural pixels are shown to have better resolution than those that use the filtered backprojection algorithm.

I. INTRODUCTION

The concept of generalized natural pixels has arisen in the search for methods to reduce image artifacts in the reconstruction of truncated projections as well as to reduce the complexity of the backprojection operator in iterative reconstruction algorithms used in single photon emission computed tomography (SPECT). Research [1–3] has shown that the accuracy and precision of projection reconstruction can be improved for cases of truncated and attenuated projections by using a natural pixel basis rather than a square pixel image basis. In these cases the natural pixels were defined as strips corresponding to the imaging geometry such that both the projection and backprojection operation were based upon the same set of natural pixels. This paper defines the backprojection based upon a set of natural pixels differing from that which the projection operation is based upon. Formulation of the reconstruction problem using generalized natural pixels offers improved image quality in the reconstruction of truncated projections, and retains the flexibility of modeling the physics of the image detection process while reducing the complexity of the algorithm operations, namely the calculation of the backprojection operation during each iterative step.

Backprojection and projection can be defined with different natural pixel bases because a least squares approach is used to solve the reconstruction problem. The reconstruction is formulated as a solution to the imaging equation \( Mq = FGq = p \), wherein \( p \) is the projection measurement, \( q \) is the unknown vector, \( F \) is the projector, and \( G \) is a generalized backprojector defined so that the reconstructed image is an expansion of natural pixels with coefficients equal to the components of the solution \( q \). It has been common practice to define the backprojection operator \( G \) as the adjoint of the projection operator. However, \( G \) can be defined quite arbitrarily, thus allowing the vector space for the solution \( q \) to differ from that of the projection measurements \( p \). For example, if \( F \) is a fan beam projection operator, \( G \) can be a parallel backprojection operator; or if \( F \) is a truncated projection operator, \( G \) can be formulated as a non-truncated backprojection operator. Using an algorithm that inverts the equation \( Mq = FGq = p \), such as the conjugate gradient algorithm or the singular value decomposition (SVD) method, a different solution for \( q \) can be determined for different choices of the backprojector \( G \). Assuming the image \( \rho \) can be expanded using a generalized natural pixel basis, the final image is obtained by backprojecting \( q : \rho = Gq \), using these generalized natural pixels, and sampling \( \rho \) at a discrete array of points.

The generalized natural pixel is a generalization of the natural pixel representation. If photon scatter, geometric...
response, and attenuation are ignored, a natural pixel basis arises “naturally” from the scanning geometry without digitizing the image [4-6]. The basis forms a decomposition of the image plane into a set of overlapping pixels, which are strips uniquely defined by the paths of the projection rays. The integral of the activity distribution over a natural pixel is the projection of the activity within a particular strip into a projection bin. Under this formulation, it has previously been considered [1-10] that the image is represented as the backprojection of a vector along the same natural pixels used in defining the projections. By using a generalized natural pixel basis, the image representation in the backprojection need not be the same as in the projection model. The reconstruction problem can be formulated as determining a solution from a vector space differing from that of the projection measurements.

This study uses parallel beam and fan beam geometries to investigate the reconstruction of projections that utilize generalized natural pixels. In the first example a non-truncated parallel beam backprojection operator is used to reconstruct truncated parallel projection measurements. In the second, fan beam and parallel beam backprojection operators are used to reconstruct fan beam projections. The fan ray is approximated by a parallel strip with center line collinear along a line from the center of the projection bin to the focal point. The bin width, the number of projection bins, and the number of projection views of the backprojection operator are varied to study the behavior of the “generalized natural pixels” in truncation and non-truncation tomography.

II. THEORY

A. Generalized Natural Pixel Representation

A natural pixel basis is a decomposition of the x-y plane into a set of pixels defined by the paths of the projection rays for a specified geometry. Each path corresponds to a natural pixel, which forms the support of a characteristic function. The collection of all the characteristic functions forms the natural pixel basis set. An example of a natural pixel corresponding to the angle index \( m \) and the projection bin index \( j \) is shown in Fig. 1 with the characteristic function \( \chi_{jm}(\xi) \), where \( \xi \) is a vector in the x-y plane. The value of the characteristic function is one for \( \xi \) in the strip corresponding to the natural pixel, and zero otherwise.

In this paper one natural pixel basis, the set of characteristic functions \( \{\chi_{jm}(\xi)\} \), corresponds to the path of projection rays. Another natural pixel basis, the set of characteristic functions \( \{\Psi_{jm}(\xi)\} \), corresponds to the path of backprojection rays. The basis set \( \{\Psi_{jm}(\xi)\} \) is said to be a generalized natural pixel representation in that the ray paths need not be the same as the measured projection rays.

The projection operator \( F \) is a mapping from the space of continuous functions \( p \) to a vector space of vectors \( p \) of projection measurements [11,12]. The operator \( F \) is a vector whose components are functionals: \( F = (F_{jm}) \). Each functional \( F_{jm} \) maps the continuous function \( p \) to a measured projection value \( F_{jm}(p) \) by integrating the function \( p \) over the natural pixel that corresponds to the angle index \( m \) and the projection bin index \( j \). The functional \( F_{jm} \) for the projection transform \( F \) is defined as:

\[
F_{jm}(p) = \int_D \chi_{jm}(\xi)p(\xi)d\xi = p_{jm},
\]

where the integral over \( \xi \) is restricted to a finite support \( D \), and the characteristic function \( \chi_{jm}(\xi) \) has a support corresponding to the natural pixel with indices \( j, m \). Thus the projection operation maps the continuous function \( p \) into a vector \( p \) of measurements \( p_{jm} = F_{jm}(p) \).

The basis set \( \{\chi_{jm}(\xi)\} \) is commonly referred to as a natural pixel basis set because it is usually thought to arise naturally from the detection geometry, which in x-ray CT is the integral of the continuous attenuation distribution over narrow ray paths. In SPECT the physics of the image detection process is much more complicated and needs to be included in the

![Figure 1. Parallel Geometry. A natural pixel is illustrated as the support for the characteristic function \( \chi_{jm}(\xi) \) corresponding to the angle index \( m \) and the projection bin index \( j \). The value of the characteristic function is one for \( \xi \) inside the strip in the x-y plane; thus, the integral of \( p(\xi) \) over the natural pixels within the support \( D \) maps the continuous function \( p \) into a vector of measurements \( p_{jm} \).](image-url)
This paper does not consider this aspect of the physics, but concentrates more on the geometric aspects of fan beam and parallel beam detector geometry.

Next we consider the backprojection operator $G$, which maps vectors from a vector space of discrete projection values to continuous image functions. The backprojection operator $G$ operates on a vector $q$ with elements $q_f$ giving:

$$(Gq)(r) = \sum_m \sum_f \Psi_{jm}(r) q_f,$$  

where the characteristic functions $\Psi_{jm}(r)$ used in the backprojection operation have supports corresponding to natural pixels which may be different than those used in Eq. (1) to define the projection operation. Thus, $G$ maps a projection vector into a continuous function so that the value at the position $r$ in Fig. 2 is the summation of the elements $q_f$ times the corresponding characteristic functions $\{\Psi_{jm}(r)\}$.

Before proceeding, let's look at an example of where natural pixels used to define the backprojection operation differ from those that define the projection operation. One natural pixel basis may be used to define a fan beam projector in conjunction with another used to define a parallel backprojector. In this formulation, the backprojection is not the adjoint of the projection, but is allowed to take on various generalized forms.

Combining the expressions for the projection operation in Eq. (1) and the backprojection operation in Eq. (2), we see that the backprojection-projection operator gives a real number for each functional $F_{jm}$:

$$F_{jm}(Gq) = \sum_m \sum_f d_{jm}(\xi) \Psi_{jm}(r) d\xi.$$  

In matrix form the backprojection-projection operator is:

$$Mq = p,$$  

where each element of $M$ is given by the integral in Eq. (3):

$$M_{jm} = \int_D \Psi_{jm}(r) d\xi.$$  

The matrix equation in Eq. (4) can be solved for $q$ using an iterative algorithm or inverting the matrix $M$ using an SVD method. The reconstructed image is obtained by sampling the backprojection of $q$ in Eq. (4) at a discrete array of points.

B. Natural Pixels for Parallel Beam Geometry

If we do not consider the detection of scattered photons, photon attenuation, and system geometric response, the projection measurement $p_{jm}$ for the parallel beam geometry in Fig. 1 will count all photons within a strip bounded by two parallel lines. This strip is a natural pixel shown in Fig. 2 to have a width $w_p$ with a center line perpendicular to the detector at the center $\xi_{pj}$ of the projection bin. The natural pixel basis is the set of characteristic functions $\{\chi_{jm}(r)\}$ with natural pixels as supports:

$$\chi_{jm}(r) = \begin{cases} 1 & (\xi_{pj} - w_p/2) \leq \xi < (\xi_{pj} + w_p/2) \\ 0 & \text{otherwise} \end{cases},$$  

where

$$\xi = r \cdot \theta_m.$$  

The unit vector $\theta_m$ is parallel to the face of the detector (Fig. 2).

C. Natural Pixels for Fan Beam Geometry

Figure 3 illustrates the fan beam geometry. For the projections the focal length is denoted as $F$ and the distance between the detector and the center of rotation is $R$. It is assumed that the projection bin for the fan beam geometry accepts all photons within a strip also bounded by two parallel lines, which parallel the line from the focal point to the center $\xi_{pj}$ of the projection bin. The natural pixels are the collection of these swaths of parallel rays of width:

$$width_{pj} = w_p F / (\sqrt{F^2 + \xi_{pj}^2}).$$  

Also, note that the width of the natural pixel depends upon the center of the projection bin $\xi_{pj}$. 

Figure 2. Parallel Geometry. Natural pixels are shown with characteristic functions $\chi_{jm}$ and $\Psi_{jm}$ for the projection and backprojection operations, respectively. The projection $p_{jm}$ is rotated by $\theta_m$ with unit vector $\hat{\theta}_m$ parallel to the detector, and $\alpha_{jm}$ is the angle difference between the estimation $q_m$ and the projection $p_{jm}$. The center of the projection bins for $p_{jm}$ and $q_m$ are $\xi_{pj}$ and $\xi_{q}$, respectively. The bin widths for $p_{jm}$ and $q_m$ are $w_p$ and $w_q$, respectively.
The natural pixel basis for the fan beam geometry is the set of characteristic functions \{ \chi_{jm}(\xi) \} with these natural pixels as support:
\[
\chi_{jm}(\xi) = \begin{cases} 
1 & \text{if } (\xi_p^m - \wp/2) \leq \xi < (\xi_p^m + \wp/2) \\
0 & \text{otherwise} 
\end{cases}, \quad (8)
\]
where
\[
\xi = \frac{R - (\xi \cdot \vec{\theta}_m)}{F} \xi_p^m + \xi \cdot \vec{\theta}_m. \quad (9)
\]
The unit vector \( \vec{\theta}_m \) is shown in Fig. 3 to parallel the detector plane; the unit vector \( \vec{\theta}_m^{-1} \) is orthogonal to \( \vec{\theta}_m \). Note the similar form as in Eq. (6), but \( \xi \) in Eq. (9) differs from that in Eq. (7).

D. Backprojector-Projector Operator - The M Matrix

This section derives expressions for the elements of the matrix \( M \) in Eq. (5) for three generalized pixel geometries.

1. Parallel-Beam/Parallel-Beam

First consider a geometry that has parallel rays for both the projection and backprojection operations. The natural pixel basis for the projection operation is given in (6). The parallel geometry backprojection operator has a natural pixel basis of characteristic functions \{ \Psi_{jm}(\xi) \} where the characteristic function \( \Psi_{jm}(\xi) \) is defined as:
\[
\Psi_{jm}(\xi) = \begin{cases} 
1 & \text{if } (\xi_p^m - \wp/2) \leq \xi < (\xi_p^m + \wp/2) \\
0 & \text{otherwise} 
\end{cases}, \quad (10)
\]
where
\[
\xi_p^m = \xi \cdot \vec{\theta}_m'. \quad (11)
\]

When the intersection of two natural pixels is completely within the support \( D \), as shown in Fig. 2, an analytical expression for the elements of the matrix \( M \) can be derived by rewriting Eq. (5) for characteristic function \( \chi_{jm}(\xi) \) defined in Eq. (6) and characteristic function \( \Psi_{jm}(\xi) \) defined in Eq. (10) as
\[
M_{jm} = \int \int d\xi d\zeta \psi_{jm}(\xi) \chi_{jm}(\xi), \quad (12)
\]
where
\[
\psi_{jm}(\xi) = (\xi_p^m + \wp/2) \csc \alpha_m - \xi \cot \alpha_m, \quad (13)
\]
and
\[
\xi = (\xi_p^m - \wp/2) \csc \alpha_m - \xi \cot \alpha_m. \quad (14)
\]

Figure 3. Fan Beam Geometry. Natural pixels bounded by parallel rays of constant width are shown with characteristic functions \( \chi_{jm} \) and \( \Psi_{jm} \) for the fan beam projection and backprojection operations, respectively. The projection \( p_m \) with focal length \( F \) and radius of rotation \( R \) is rotated by \( \theta_m \) with unit vector \( \vec{\theta}_m \) parallel to the detector, and \( \alpha_m \) is the angle difference between the estimation \( q_m \) and the projection \( p_m \). The geometry for the estimation \( q_m \) has focal length \( F' \) and a radius of rotation \( R' \). For \( p_m \) and \( q_m \), the center of the projection bins are \( \xi_p^m \) and \( \xi_q^m \), and the bin widths are \( \wp \) and \( \wp_q \), respectively. The widths \( \text{width}_p \) and \( \text{width}_q \) of the natural pixels are equal to the width of the projection bin times the sine of the angle between the rays of the natural pixel and the detector.

Figure 4. For parallel projection and parallel backprojection operations, the entries in the matrix \( M \) are determined by integrating Eq. (12) over the shaded area within the support \( D \).
The lines \( \zeta_1(\xi) \), and \( \zeta_2(\xi) \) are shown in Fig. 4. The angle \( \alpha_{m'm} \) is the difference between the angles: \( \alpha_{m'm} = \theta_{m'} - \theta_m \).

The integral in Eq. (12) can be expressed analytically as

\[
M_{jm}^{jm} = \frac{w_p w_q}{\sin \alpha_{m'm}}. \tag{15}
\]

The width of each natural pixel need not be the same. Therefore, a more general formulation would be

\[
M_{jm}^{jm} = \left| \frac{\text{width}_{pj} \cdot \text{width}_{qj}}{\sin \alpha_{m'm}} \right|, \tag{16}
\]

where the width of a natural pixel depends upon the projection bin, which is labelled with indices \( p, j \) and \( q, j' \). When the overlapping area falls partially within the contour support \( D \), the area is calculated by adding the areas of a polygon and the smaller areas bounded by the support as shown in Fig. 5. When \( \sin \alpha_{m'm} = 0 \), two natural pixels are parallel. The element of the matrix \( M \) is the overlapped area of these two parallel natural pixels contained within the support \( D \).

2. Fan-Beam/Parallel-Beam

Next consider a combination of fan beam geometry for the projections and parallel beam geometry for the backprojection. The natural pixel basis for the projection operation is given in Eq. (8) and for the backprojection in Eq. (10). To derive an expression for the intersection of two natural pixels, first the fan beam projection angle \( \theta_m \) will be transformed to the parallel projection angle \( \beta_{m'}^j \) as shown in Fig. 6. (Note that this transformation depends upon the location of the projection bin \( j \).) Then the sine of the difference between the parallel projection angles, \( \sin(\theta_{m'} - \beta_{m'}^j) \), will be calculated and this will be substituted for the expression of \( \sin \alpha_{m'm} \) in Eq. (16).

The transformation between fan beam projection angle \( \theta_m \) and parallel projection angle \( \beta_{m'}^j \) is

\[
\beta_{m'}^j = \theta_m + \arctan\left(\frac{\zeta_{pj}^2}{F}\right). \tag{17}
\]

The sine of the difference between the two parallel projection angles is

\[
\sin(\theta_{m'} - \beta_{m'}^j) = 
\frac{(\sin \alpha_{m'm})^2 F^2 + \zeta_{pj}^2}{(\sin \alpha_{m'm})^2 F^2 + \zeta_{pj}^2} - (\cos \alpha_{m'm})^2 \frac{\zeta_{pj}^2}{(\sin \alpha_{m'm})^2 F^2 + \zeta_{pj}^2}, \tag{18}
\]

where \( \alpha_{m'm} = \theta_{m'} - \theta_m \) is the difference between the projection angles, one being the projection angle for parallel geometry and the other for fan beam. The width between the parallel rays of the natural pixel is

\[
\text{width}_{pj} = w_p \frac{F}{(\sin \alpha_{m'm})^2 + \zeta_{pj}^2}, \tag{19}
\]

where \( w_p \) is the width of the bin on the face of the detector. Substituting \( \sin(\theta_{m'} - \beta_{m'}^j) \) for \( \sin \alpha_{m'm} \) in Eq. (16) gives

\[
M_{jm}^{jm} = \left| \frac{\text{width}_{pj} \cdot \text{width}_{qj}}{\sin(\theta_{m'} - \beta_{m'}^j)} \right|, \tag{20}
\]

which after substituting the expression for \( \text{width}_{pj} \) in Eq. (19), \( \text{width}_{qj} = w_q \), and the expression for \( \sin(\theta_{m'} - \beta_{m'}^j) \) in Eq. (18) into Eq. (20) gives

\[
M_{jm}^{jm} = \left| \frac{F w_p w_q}{F \sin \alpha_{m'm} - \zeta_{pj} \cos \alpha_{m'm}} \right|. \tag{21}
\]
If $F \sin \alpha_{m'n'} \cdot \xi_{pj} \cos \alpha_{m'n'} = 0$, two natural pixels are parallel. The element of the matrix $M$ is the overlapped area of these two natural pixels contained within the support $D$.

3. Fan-Beam/Fan-Beam

The last case considered is one in which both the projection and backprojection geometries are fan beam, as shown in Fig. 3. To derive an expression for the intersection of two natural pixels, the fan beam projection angles $\theta_m$ and $\theta_{m'}$ will be transformed to parallel projection angles $\beta^j_m$ and $\beta^j_{m'}$. Then the sine of the difference between the parallel projection angles, $\sin(\beta^j_{m'} - \beta^j_m)$, will be calculated and will be substituted for the expression of $\sin(\alpha_{m'n'})$ in Eq. (16).

The transformation between fan beam projection angle $\theta_m$ and parallel projection angle $\beta^j_m$ is

$$\beta^j_m = \theta_m + \arctan(\xi_{pj}/F),$$

and the transformation between fan beam projection angle $\theta_{m'}$ and parallel projection angle $\beta^j_{m'}$ is

$$\beta^j_{m'} = \theta_{m'} + \arctan(\xi_{qj}/F).$$

The sine of the difference between the two parallel projection angles is

$$\sin(\beta^j_{m'} - \beta^j_m) =$$

$$\sin(\theta_{m'} - \theta_m) \cos(\arctan(\xi_{qj}/F) - \arctan(\xi_{pj}/F))$$

$$+ \cos(\theta_{m'} - \theta_m) \sin(\arctan(\xi_{qj}/F) - \arctan(\xi_{pj}/F)),$$

where

$$\cos(\arctan(\xi_{qj}/F) - \arctan(\xi_{pj}/F)) =$$

$$(F/(\sqrt{F^2 + \xi_{qj}^2}))(F/(\sqrt{F^2 + \xi_{pj}^2}))$$

$$+ (\xi_{qj}/(\sqrt{F^2 + \xi_{qj}^2}))(\xi_{pj}/(\sqrt{F^2 + \xi_{pj}^2})),$$

and

$$\sin(\arctan(\xi_{qj}/F) - \arctan(\xi_{pj}/F)) =$$

$$(\xi_{qj}/(\sqrt{F^2 + \xi_{qj}^2}))(\xi_{pj}/(\sqrt{F^2 + \xi_{pj}^2}))$$

$$- (F/(\sqrt{F^2 + \xi_{qj}^2}))(\xi_{pj}/(\sqrt{F^2 + \xi_{pj}^2})).$$

The width between the parallel rays of the natural pixels are

$$\text{width}_{pj} = w_p F/(\sqrt{F^2 + \xi_{pj}^2})$$

and

$$\text{width}_{qj} = w_q F/(\sqrt{F^2 + \xi_{qj}^2}).$$

where $w_p$ and $w_q$ are the widths of the bins on the respective detectors. Substituting $\sin(\beta^j_{m'} - \beta^j_m)$ for $\sin(\alpha_{m'n'})$ in Eq. (16) gives

$$M'^{m'}_{m''} = \frac{\text{width}_{pj} \cdot \text{width}_{qj}}{\sin(\beta^j_{m'} - \beta^j_m)}.$$

which after substituting the expression for $\text{width}_{pj}$ in Eq. (27), substituting the expression for $\text{width}_{qj}$ in Eq. (28), and the expression for $\sin(\beta^j_{m'} - \beta^j_m)$ in Eq. (24) into Eq. (29), gives

$$M'^{m'}_{m''} =$$

$$\frac{F w_p F' w_q}{(F F' + \xi_{pj}^2 \xi_{qj}^2) \sin(\alpha_{m'n'}) + (F F' \xi_{qj}' - F F' \xi_{pj}') \cos(\alpha_{m'n'})}.$$
Table I lists the parameters used to generate the truncated projections. The non-truncated projections (type \( p_1 \)) had 40 projection bins. For each phantom study the projection data type \( p_2 \) and type \( p_3 \) were generated from the projection data type \( p_1 \) by truncating each projection from 40 bins into 30 and 20 bins to simulate different degrees of truncation. In the simulated projection data the value of each projection bin was calculated by summing across the projection bin 1000 line integrals of the phantom to simulate the integral across a natural pixel. No noise was simulated in any of the computer-generated projections.

A Hoffman brain phantom was used to evaluate the reconstruction of actual data collected using a triple-detector SPECT system. The phantom was injected with 20 mCi of \( ^{99m} \text{Tc} \); projection data were collected using the PRISM 3000 (Picker International, Cleveland, OH) with a high resolution parallel collimator. A total of 120 64x64 projections of 190 seconds per view were acquired over 360° from the three detectors.

First the projections were corrected for photon decay, then one slice from each projection was used to generate projection data for 64 projection bins. The 40-bin projections (type \( p_1 \)) were first generated by deleting 24 bins from the 64 bins (leaving 20 bins on either side of the center of rotation) in each projection view. The total counts summed over 180° of the 40-bin projections for the slice equalled to 9.7×10⁶. Then the 30- and 20-bin projections (type \( p_2 \) and type \( p_3 \)) were generated by deleting 10 and 20 bins respectively, from each of the 40-bin projections. Only views 31 to 90 were used for the 180° reconstructions for the parallel geometry.

The projections of the first computer-simulated circular discs were not truncated for all three types of projections. For the second torso phantom, the maximum truncation for the Hoffman brain phantom study was 0% for the 40 projection bins, 9.1% for 30 bins, and 39.4% for 20 bins.

The \( \mathbf{M} \) matrix was generated for nine different combinations of projection and backprojection geometries, which are listed in Table II. The identifiers used in the table, for example, PPM 3646, refers to parallel projection geometry (first letter), parallel backprojection geometry (second letter), the matrix \( \mathbf{M} \) (third letter) with 30 projection bins for \( p \) (first numeral), 60 projection angles (second numeral), 40 estimation bins for \( q \) (third numeral), and 60 backprojection angles (fourth numeral). In two cases the projection bin width for \( q \) was greater than 1. The supports used in forming the matrix \( \mathbf{M} \) were always larger than the non-truncated image.

The singular value decomposition (SVD) [2] was used to solve for \( q \) in the equation \( \mathbf{M}q = \mathbf{p} \). The choice of the number of singular terms was determined by observation of the

### Table I. Parallel Geometry Studies.

<table>
<thead>
<tr>
<th>geometry</th>
<th>number of bins/view</th>
<th>number of projection views / 180°</th>
<th>bin width</th>
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<tr>
<td>( p_1 )</td>
<td>parallel beam</td>
<td>40</td>
<td>10.7 mm</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>parallel beam</td>
<td>30</td>
<td>10.7 mm</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>parallel beam</td>
<td>20</td>
<td>10.7 mm</td>
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</table>

### Table II. Parallel Geometry Studies.

<table>
<thead>
<tr>
<th>geometry</th>
<th>parallel beam - ( p ) vector</th>
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<tbody>
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<tr>
<td>number views/180°</td>
<td>60</td>
</tr>
<tr>
<td>bin width (10.7mm)</td>
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<tr>
<td>geometry</td>
<td>parallel beam - ( q ) vector</td>
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<td>number bins/view</td>
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<tr>
<td>image size</td>
<td>40x40</td>
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</table>
reconstructions for different numbers of singular terms. The number that gave the best trade-off between resolution and noise was selected. The final images were obtained by backprojecting \( q \) to a discrete array of points comprising a grid of 40x40.

B. Generalized Natural Pixels Applied to the Reconstruction of Fan Beam Projections

The physical Hoffman brain phantom was used to acquire the fan beam projection data. The phantom was injected with 20 mCi of \(^{99}\text{Tc}\); projection data was collected using the Picker PRISM 3000 with a high resolution fan beam collimator of 65 cm focal length. A total of 120 64x64 projections of 150 seconds per view were sampled over 360° from three detectors. First the projections were corrected for photon decay, then one slice from each projection was used to generate projection data for 64 projection bins. The projections with 32 projection bins used in the study was generated by combining two projection bins into one bin. Only the even number of views were used in the reconstruction. The total counts summed over the projection data with 32 projection bins and 60 views was equal to \( 6.2 \times 10^6 \).

The \( M \) matrix was generated for three different combinations of projection and backprojection geometries. The parameters for the geometries are given in Table III. One geometry used fan beam for both the projection and the backprojection operators with the same number of projection angles, bin widths, and number of bins for each. The other two geometries used parallel beam backprojectors. In one case the bin width was 10.7 mm; in the other, 7.3 mm. The width of 7.3 mm was determined equal to \( (F-R)/F \) times the projection bin width of 10.7mm, wherein \( F \) is the focal length and \( R \) is the distance from the detector to the center of rotation. This insured that even with the smaller bin width the 32 natural pixels representing \( q \) would still extend over the entire phantom.

Fan beam projections were reconstructed using the SVD method to solve for \( q \) in the equation \( Mq = p \). The final images were obtained by backprojecting \( q \) to a discrete array of points comprising a grid of 64x64. These SVD reconstructions were compared with a reconstruction using the filtered backprojection algorithm [13].

IV. RESULTS

A. Generalized Natural Pixels Applied to the Reconstruction of Truncated Parallel Projections

The reconstructions of truncated and non-truncated projections using various geometries for \( q \) listed in Table II are shown in Figs. 7, 8, and 9. Figure 7 shows the reconstructions without truncation. Figures 8 and 9 show reconstructions for projection type \( p_2 \), and projection type \( p_3 \), respectively. In the truncation studies, \( q \) is selected to have either more projection bins than \( p \) or a larger bin width so that it extends beyond the truncation edge with an extent large enough to include the projection of the entire object.

In studies PPM3646 and PPM2646, \( q \) had a bin width equal to the projection bin width of \( p \), but with more bins in order to include those bins not measured outside the truncation edge. This reconstruction approach introduces smearing outside the image, but reduces the edge ring artifact and preserves resolution within the non-truncated region. Whereas in studies PPM3636p and PPM2626p, \( q \) had the same number of bins as projection bins but a larger bin width so that \( q \) included the entire activity distribution. The reconstructions also have smearing outside the image and reduced edge ring artifacts.
Figure 7. Reconstructions of parallel projections of type \( p_i \) using generalized natural pixels. All results were obtained via the singular value decomposition of a 2400x2400 \( M \) matrix. The inversion of \( M \) was obtained by using 990 singular terms. All images are displayed on a 40x40 grid. (See Tables I and II for reconstruction parameters).

Figure 8. Reconstructions of parallel projections of type \( p_2 \) using generalized natural pixels. All results were obtained via the singular value decomposition of the matrix \( M \). The sizes of the matrix \( M \) were 1800x1800 (Column 1), 1800x2400 (Column 2), 1800x1800 (Column 3), 1800x1200 (Column 4). The singular terms used for the inversion of \( M \) are listed in parenthesis. All images are displayed on a 40x40 grid. The last “p” in the identifier PPM3636p refers to a larger bin width than 1 (10.7 mm).
Figure 9. Reconstructions of parallel projections of type $p_3$ using generalized natural pixels. The sizes of the matrix $M$ were 1200x1200 (Column 1), 1200x2400 (Column 2), 1200x1200 (Column 3), 1200x1200 (Column 4). The singular terms used for the inversion of $M$ are listed in parenthesis. All images are displayed on a 40x40 grid. The last “p” in the identifier PPM2626p refers to a larger bin width than 1 (10.7 mm).
Because the bin width of $q$ is wider, the resolution of the reconstruction deteriorates. To maintain the number of undetermined variables while increasing the number of projection bins for $q$, studies of PPM3643 and PPM2643 increased the number of estimation bins per projection but reduced the number of views. This gave less or the same number of unknowns as in the sampled projections $p$. The results show that edge ring artifacts are reduced, but smearing is seen outside the image, and resolution deteriorates. In general, for truncated projections the estimation of $q$ that covers the extended projection of the object is effective in reducing ring artifacts (PPM3646, PPM2646, PPM3643, PPM2643, PPM3636p, PPM2626p); however, using more projection bins (PPM3646, PPM2646, PPM3643, PPM2643) is much more effective in preserving the resolution than is increasing the projection bin width (PPM3636p, PPM2626p).

Notice that with the larger projection bin width for $q$, the number of singular terms used for the reconstruction of PPM3636p and PPM2626p were significantly smaller than for the other reconstructions. For this case, increasing the number of singular terms significantly increased both noise and artifacts in the reconstruction.

B. Generalized Natural Pixels Applied to the Reconstruction of Fan Beam Projections

The reconstructions of fan beam projections from the Hoffman brain phantom using various geometries for $q$ listed in Table III are shown in Fig. 10. The results are shown using, in one case, a fan beam backprojector with natural pixels bounded by parallel rays separated by a width given in Eq. (28), and, in the other two cases, using a parallel backprojector with natural pixels, one with parallel rays separated by a width equal to the projection bin width and the other by a width equal to $(F-R)/F$ times the projection bin width.

The results show that fan beam reconstructions using natural pixels with fan beam backprojector and fan beam backprojector gave higher resolution than that of the filtered backprojection reconstruction. Reconstructions using natural pixels with a fan beam projector and a parallel backprojector also gave higher resolution if the pixel width for the backprojector is selected small enough. The parallel backprojector with natural pixel width equal to the projection bin gave a reconstruction with the worse resolution of all, whereas a natural pixel with a smaller width gave results equivalent to that obtained using the fan beam backprojector.

V. DISCUSSION

A linear least squares method was used to reconstruct fan beam, truncated and non-truncated parallel projection measurements that utilized backprojection geometries different from the projection geometries. A generalized natural pixel basis was selected to model various geometries in the backprojector for a given projector model. The reconstruction of truncated projections shows that estimating the solution vector with enough projection bins to cover the non-truncated projection of the object is effective in reducing the edge ring artifact. Reconstruction of fan beam projections acquired from a SPECT system also shows that a generalized natural pixel basis approximates the geometric response of the collimator, thus fan beam reconstructions are shown to have better resolution than those that use the filtered backprojection algorithm.

Natural pixels bounded by parallel rays of constant width were used to model the fan beam projections. To our knowledge this has never been implemented before. This model was used for two reasons: To easily calculate the intersection of overlapping natural pixels, and to approximate, at least to first order, the geometric spread of the collimator holes. If fan beam strips [14], which diverge from a focal point at the focus so that their width at the detector was equal to the width of the bin, were used instead as natural pixels, these pixels would have required a non-uniform weighting distribution so that the line integral perpendicular to the central axis of the pixel would be equal for every position along the ray in order to preserve counts. A parallel strip focused to the focal point but with equal width from the detector bin to the focal point satisfies the equal weighting requirement and is much easier for calculating the intersection of overlapping pixels. On the other hand, a parallel strip is also a good approximation for fan beam geometry; it better represents the physics of the collimated detector used in SPECT in that a collimator hole does not see rays converging to a focal point at some distance from the detector, but, because of the finite aperture, sees rays that diverge from the detector with focus at the detector, much like an inverted fan beam geometry. Therefore using uniform parallel strips as natural pixels for fan beam geometry acknowledges the discrete form of the measurements by using a geometrically realistic approach, and makes it easier to calculate the elements of the matrix $M$.

This approach for modeling the fan beam geometry with natural pixels resulted in reconstructions with better resolution
Figure 10. Reconstructions of fan beam projections via generalized natural pixels. Three different combinations of the matrix $\mathbf{M}$ (see Table III for parameters) are compared with the filtered backprojection algorithm. The top left diagram shows the geometry of the projection $p$. The three diagrams below show the geometry of $q$ corresponding to the reconstructed image. All images are displayed on a 64x64 grid.
than those produced by the filtered backprojection algorithm for physically acquired data. For the filtered backprojection algorithm, the measurements are assumed continuous, and the reconstruction is formed after filtering by backprojecting along rays from the center of the detector bin that pass through the focal point. This procedure ignores the discrete properties of the acquired measurements and does not properly model the geometric spreading of the collimated detector. Therefore the resolution is lower than a reconstruction approach that uses a natural pixel basis.

Three different backprojection geometries were used to reconstruct fan beam projections. In the first a fan beam geometry was used for the backprojection with natural pixels bounded by parallel rays of constant width equal to the projection bin width. In the second a parallel geometry was used for the backprojection with natural pixels of width equal to the projection bin width. In the third a parallel geometry was used for the backprojection with a natural pixel of width equal to \((F-R)/F\) times the projection bin width. The third study produced a resolution almost equal to that obtained with the fan beam backprojector in the first study. It would appear that a smaller pixel width is necessary for a parallel beam backprojector to achieve the same resolution as that produced by a fan beam backprojector with natural pixels bounded by parallel rays wide enough to encompass the projection bin at the detector.

The uniform weighted strips bounded by parallel rays, used for the generalized natural pixel basis to reconstruct both fan beam and parallel projections, still may not be the optimal basis to represent a continuous image. For example, strip structure artifacts are seen if an estimated \(q\) of 32 bins for 64 views over 360° is backprojected into a 256x256 grid. Linear interpolation helps remove these strip artifacts; we believe that a natural pixel with a triangular weighted strip would be superior to a strip with constant weighting for the estimation of large image arrays from a few projection samples. Although it is more complicated and computer intensive to implement a reconstruction algorithm with this basis, it should produce a better reconstruction. The triangular weighted natural pixels is an example of another class of generalized natural pixels.

The concept of the generalized natural pixel basis can be used to model other system and physical factors, such as attenuation and geometric detector response. Previously natural pixels have been used to reconstruct exponential and variable attenuated Radon projections [2, 3], in which the transpose of the natural pixel projection operator were used as the backprojector. For the variable attenuation case, the implementation was rather complicated in forming the matrix \(M\) [3]. To improve efficiency, backprojection operator that does not model the attenuation could be used; a uniformly weighted natural pixel could be used instead for the backprojection, making it much easier to calculate the elements of the matrix \(M\). Note that the final image would be formed by using this uniformly weighted natural pixel backprojector. The same approach could be used to model and correct for the system geometric response. Therefore generalized natural pixels can be used to correct for attenuation or collimator geometric response by using projectors that model attenuation or geometric response and by using backprojectors that do not.

Reconstructions produced via a generalized natural pixel basis show ring artifacts of less amplitude than do reconstructions of truncated projections that use the filtered backprojection algorithm. The characteristics of natural pixel reconstruction is very different than that of filtered backprojection reconstruction. First, the ramp filter used in tomography enhances the truncation edge and accentuates the ring artifact. Second, the filtering is applied to every projection independent of information about other projections, whereas reconstructions based on natural pixels or, for that matter, based on any matrix inverse approach, use information from all projection views to form the inverse solution to the reconstruction problem. Therefore artifacts caused by truncated data are reduced, as the simulation results show. The application of natural pixels to other limited data problems has also shown to reduce reconstruction artifacts, particularly when natural pixels are applied to the limited angle problem [7].

A non-weighted least squares approach was implemented in this paper to verify the potential use of generalized natural pixel basis. A weighted least squares solution can be obtained by including statistical weights in the matrix \(M\) [10]; this solution is expected to give more efficient estimates of reconstructions from noisy projections. An even better approach would be to model the statistics as a Poisson distribution, which is more appropriate for gamma camera counting. Algorithms could also be designed to include prior information in the solution set. These aspects of generalized natural pixel reconstruction still need to be investigated.

Generally for a 2400x2400 \(M\) matrix used in our simulations, the number of zero entries equaled 2.61768 x 10^6, which gave a sparsity of approximately 45%. The calculation of the matrix \(M\) required only 8 seconds of computation time.
on a SUN SPARCstation 2000 (40 MHz super SPARC processor, 1 Gbyte of memory). It then required 3.75 hours to perform the SVD of $M$. In our previous paper [3], we calculated $M$ for a variable attenuated backprojector-projector. The calculation of $M$ for this case required 24 hours of computation time on the same processor with 256 Mbytes of memory instead of the 1 Gbyte. It then required 6 hours of CPU time to perform the SVD of $M$. If one models the variable attenuation, it requires considerably more time to calculate the $M$ matrix. This time can be improved with a backprojector of generalized pixels, which does not model the attenuation.

The flexibility of choice in the generalized natural pixels used in the backprojector can reduce the complexity of forming the matrix $M$ and improve the reconstruction quality, as seen in the reconstruction of truncated projections. It is conceivable that backprojection operators with any arbitrary geometry and with varied numbers of bins and various bin widths could be used to reconstruct any detection geometry. Attenuation, geometric response, and scatter also may be included in the projection operator and excluded from the backprojection operator. Moreover, a reconstructed image may be represented by generalized natural pixels other than uniform strips, such as natural pixels with triangular weighted strips that impose an interpolation. Therefore a generalized natural pixel basis may be very general. How general is a question that needs further investigation.

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VII. REFERENCES


