A Reconstruction Algorithm Using Singular Value Decomposition of a Discrete Representation of the Exponential Radon Transform Using Natural Pixels

G. T. Gullberg and G. L. Zeng

Abstract—An algorithm to correct for constant attenuation in SPECT is derived from the singular value decomposition (SVD) of a discrete representation of the exponential Radon transform using natural pixels. The algorithm is based on the assumption that a continuous image can be obtained by backprojecting the discrete array $q$, which is the least squares solution to $Mq = p$, where $p$ is the array of discrete measurements, and the matrix $M$ represents the composite operator of the backprojection operator $A^*_n$ followed by the projection operator $A_n$. A singular value decomposition of $M$ is used to solve the equation $Mq = p$, and the final image is obtained by sampling the backprojection of the solution $q$ at a discrete array of points. Analytical expressions are given to calculate the matrix elements of $M$ that are integrals of exponential factors over the overlapped area of two projection strip functions (natural pixels). A spectral analysis of the exponential Radon transform is compared with that of the Radon transform. The condition number of the spectrum increases with increased attenuation coefficient, which correlates with the increase in statistical error propagation seen in clinical images obtained with low-energy radionuclides. Computer simulations using 32 projections sampled over 360 degrees show an improvement in the SVD reconstruction over the convolution backprojection reconstruction, especially when the projection data is corrupted with noise.

I. INTRODUCTION

Previously a convolution reconstruction algorithm was developed to reconstruct projections of emission sources within a constant attenuating medium [1], [2]. The algorithm was derived based on the ability to represent the projection data as projections of the exponential Radon transform; it uses an exponential backprojector after applying a reconstruction filter to the modified projection data. The major problem with the algorithm is the severe noise amplification. Subsequently we have shown that a singular value decomposition (SVD) reconstruction method is less sensitive to noise than the convolution backprojection algorithm [3, see Appendix]. The algorithm was derived from the SVD of the exponential Radon transform of a square (nonoverlapping) pixelized image model. More recently we investigated the use of a natural pixel model to improve reconstruction artifacts obtained with iterative reconstruction algorithms [4]. We found that natural pixel basis was less sensitive to noise and reconstruction artifacts than the nonoverlapping square pixel image model. This was especially evident with the reconstruction of truncated projections. The purpose of this paper is to show that the reconstruction of exponential Radon projections can further be improved using the combination of a natural pixel model and SVD reconstruction, that is, to show that using an SVD expansion with a natural pixel basis is less noise sensitive than using an SVD expansion based on the conventional square pixel image representation or using the convolution backprojection reconstruction algorithm.

Various image basis functions have arisen out of an SVD approach to solving the inverse problem [5]-[59]. These basis functions are derived from a singular value decomposition of the composite projection and backprojection (adjoint) operator. Other image representations, such as square and polar pixels, have not been developed from a mathematical structure but nonetheless have been used for convenience and speed in implementing the reconstruction algorithm, especially with iterative reconstruction algorithms [60]-[63]. In fact it was an SVD approach that was first used in computed tomography [10], [11]. A Fourier series, which incorporates a Zernike polynomial expansion for the radial coordinate, was used to evaluate the reconstructed image [10], [11], [41], [59], whereas, for iterative reconstruction algorithms, it has been common practice to represent the image as an expansion of a nonoverlapping square pixel basis. Variations of the square pixel approach have been proposed, such as a local basis of cubic B-splines [64], to model the
object. Another approach is to expand the object in an overlapping basis that is rotationally invariant [65]. Still another approach (the topic of this paper) expands the image in a natural pixel basis that arises naturally out of the geometry of beam paths used to measure the projections. This was first proposed by Buonocore et al. [66] for the application in tomography in the reconstruction of x-ray CT images.

Natural pixel basis is a decomposition of the x-y plane into a set of overlapping pixels that are strip functions uniquely defined by the path of the projection ray. The integral of the function f over the set of all natural pixels defines a mapping from an infinite-dimensional function space into a finite-dimensional vector space [67], [68]. The integral over each natural pixel basis function defines a functional F that relates to the unknown function f to the measured value g. The linear inverse problem with the discrete measurements \( g_{x_{m},\ldots} \) is to find an object f in the class \( X \) of functions such that for a set \( \{ F_{n} \} \) of linear functionals \( F_{n}(f) = g_{n}, n = 1, \ldots, N \), in the original work of Backus and Gilbert [69] the object f was the Earth model and functionals were defined that provided the total mass, the total moments of inertia, etc. In the work of Buonocore et al. [66], the f is the distribution of linear attenuation coefficients and the functionals provided the measurement from x-ray beam paths or internally emitted photons passing through the object and being detected by a finite set of detectors. The basis set \( \{ x_{m}(r) \} \) (natural pixels) represent the ray paths of finite width from which the measurements for angle m and projection bin j are generated from the data measurements. Buonocore et al. claimed that the natural pixels better model the reconstruction of the image distribution than the nonoverlapping square pixel basis because natural pixels have advantages in producing an optimal solution that incorporates the discrete form of the measurements and are applicable in limited-data situations. Later natural pixels were applied to the limited angle reconstruction problem by Garner et al. [70]. Recently Baker [7], [8] investigated the singular value decomposition of the natural pixel model and applied it to recover the point response image degradation in positron emission tomography (PET).

Using natural pixels requires that the reconstruction be formulated as a least squares solution in projection space such that the projection of the backprojection of the solution is close to the measured projections. The algorithm is based on the assumption that a continuous image can be obtained by backprojecting the discrete array \( q \), which is the least squares solution to \( Mq = p \), where \( p \) is the array of discrete measurements and the matrix \( M \) represents the composite operator of the backprojection operation \( A_{\mu}^{*} \) followed by the projection operator \( A_{\mu} \). Previously we used a conjugate gradient algorithm to solve for \( q \) [4]. In this paper an SVD approach is used to invert the matrix \( M \) instead of an iterative approach. A singular value decomposition of \( M \) is used to solve the equation \( Mq = p \), and the final image is obtained by sampling the backprojection of the solution \( q \) at a discrete array of points.

This paper first defines the exponential Radon transform, then defines functionals of the exponential Radon transform corresponding to projections along natural pixels for the two-dimensional parallel beam geometry, then defines projection and backprojection operations. Expressions are given to calculate the elements of the matrix \( M \), which is obtained from the composite of the backprojection and projection operations. An algorithm is developed to reconstruct the image by performing an SVD of the matrix \( M \). Computer simulations are performed comparing the SVD reconstruction with natural pixels and the filtered backprojection reconstruction algorithm.

II. THEORY

A. Exponential Radon Transform

The attenuated Radon transform mathematically represents the measured projections in single photon emission computed tomography (SPECT) for an ideal detector with a delta geometric response function and no detected scattered photons. As a special case of the attenuated Radon transform, the exponential Radon transform [1], [2] is defined for a constant attenuator by modifying the measured projections through a transformation which places the detector at the center of rotation.

Let \( \rho \) be the radionuclide concentration function and \( \mu \) be the attenuation coefficient distribution, then \( p \) is the measured attenuation projection data:

\[
p(\xi, \theta) = \int_{-\infty}^{\infty} \rho(\xi\theta + \zeta\theta) a(\zeta, \xi, \theta) d\zeta
\]

where

\[
a(\zeta, \xi, \theta) = \exp \left[ -\int_{\xi}^{\mu} \mu(\zeta\theta + \zeta'\theta) d\zeta' \right].
\]

The unit vector \( \theta \) is defined as \( \theta = (-\sin \theta, \cos \theta) \) and the unit vector \( \theta^{-1} \) is defined as \( \theta^{-1} = (\cos \theta, \sin \theta) \) (see Fig. 1). If \( \mu \) is constant over the convex region \( \Omega \) and zero elsewhere, as shown in Fig. 1, then the attenuation factor in (2) can be simplified as

\[
a(\zeta, \xi, \theta) = \exp \left[ -\int_{\xi}^{\mu(\xi, \theta)} \mu d\zeta' \right] = e^{-\mu(\xi, \theta)} e^{\mu(\xi, \theta)}.
\]

The point \( \beta \) in Fig. 1, which can be positive or negative, is the intersection nearest the detector of the line \( \xi - c \), \( \theta = 0 \) with the convex set \( \Omega \). If the source distribution \( \rho \) is zero outside \( \Omega \), then the projections in (1) are

\[
p(\xi, \theta) = \int_{-\infty}^{\infty} \rho(\xi\theta + \zeta\theta^{-1}) e^{\mu(\xi, \theta)} d\zeta e^{-\mu(\xi, \theta)}.
\]

Multiplying both sides of (4) by \( e^{\mu(\xi, \theta)} \) modifies the measured projections \( p(\xi, \theta) \) so that the modified projections \( g(\xi, \theta) \) are obtained from \( p \) using

\[
g(\xi, \theta) = p(\xi, \theta) e^{\mu(\xi, \theta)} = \int_{-\infty}^{\infty} \rho(\xi\theta + \zeta\theta) e^{\mu(\xi, \theta)} d\zeta = (A_{\mu} p)(\xi, \theta).
\]
Equation (6) defines a mapping $A_\mu: \rho \rightarrow g$ for the constant attenuation coefficient $\mu$.

A convolution reconstruction algorithm was developed to reconstruct exponential Radon projections [1], [2]. The major problem with the algorithm is the severe noise amplification. The algorithm uses an exponential backprojector after applying a reconstruction filter to the projections. The point spread function of the exponential backprojection is a hyperbolic cosine function [2], which makes the point spread function nonlocal and noise amplifying. Even if window functions are applied to the filters, the window functions help very little to improve the image quality in a noisy reconstruction. Here a new algorithm is derived from the SVD of the exponential Radon transform using a natural pixel image model.

### B. Natural Pixel Representation

The natural pixels are the basis set $\{\chi_{jm}(\tau)\}$ of characteristic functions shown in Fig. 2, where the vector $\tau$ is in the $x$-$y$ plane. The functions are one for $\tau$ in the strip corresponding to angle $m$ and projections bin $j$, and zero otherwise.

The functional $A_\mu^{jm}$ for the exponential Radon transform is defined as

$$A_\mu^{jm}(\rho) = \int_D \chi_{jm}(\tau) \rho(\tau) e^{i\mu \delta(\xi - \langle \theta_m \cdot \tau \rangle)} d\tau \quad (7)$$

where $\xi = \theta_m^\perp \cdot \tau$, $D$ is circular disk support in Fig. 3, and $\chi_{jm}$ is the characteristic function for the natural pixel in Fig. 2 with indices $j, m$. The backprojection operator $A_\mu$ operates on a vector $g$ giving

$$(A_\mu^{jm}g)(\xi) = \int \sum_{m} \chi_{jm}(\tau) g_{jm} e^{i\mu \xi} \cdot \delta(\xi' - \langle \theta_m \cdot \tau \rangle) d\xi' \quad (8)$$

where $\xi' = \theta_m^\perp \cdot \tau$. Therefore, the backprojection-projection gives a real number

$$A_\mu^{jm}(A_\mu^{*}g) = \int \sum_{m} \int \chi_{jm}(\tau) \chi_{jm}(\tau') e^{i\mu \delta(\xi - \langle \theta_m \cdot \tau \rangle)} \delta(\xi' - \langle \theta_m \cdot \tau \rangle) d\xi' \quad (9)$$

The matrix $M$ is defined to have elements given by the triple integral in (10):

$$M_{jm}^{\mu} = \int \int \chi_{jm}(\tau) \chi_{jm}(\tau') e^{i\mu \delta(\xi - \langle \theta_m \cdot \tau \rangle)} \delta(\xi' - \langle \theta_m \cdot \tau \rangle) d\xi' d\xi. \quad (11)$$

Integrating over $\xi$ and $\xi'$ gives

$$M_{jm}^{\mu} = \int \chi_{jm}(\tau) \chi_{jm}(\tau') e^{i\mu \delta(\xi - \langle \theta_m \cdot \tau \rangle)} d\tau. \quad (12)$$

The integral in (12) is evaluated over the shaded area in Fig. 3, which is the intersection of two natural pixels.

For the intersection of two natural pixels within the finite support $D$, the integral in (12) can be rewritten as

$$M_{jm}^{\mu} = \int_{\xi_j}^{\xi_{j+1}} \int_{\xi_j}^{\xi_{j+1}} e^{i\mu \delta(\xi - \langle \theta_m \cdot \tau \rangle)} d\xi' d\xi. \quad (13)$$

where $\xi' = \xi \cos(\theta_m - \theta_m) - \xi \sin(\theta_m - \theta_m)$ and $\xi_j, \xi_{j+1}$, are given in Fig. 2. Let $\alpha_{jm} = \theta_m - \theta_m$ and $A = \mu(1 + \cos(\theta_m))/\sin(\theta_m)$, then the integral in (13) can be expressed as

$$M_{jm}^{\mu} = \frac{(e^{A\xi_{j+1}} - e^{A\xi_j})(e^{-A\xi_{j+1}} - e^{-A\xi_j})}{A^2 \sin \alpha_{jm}}. \quad (14)$$

When the attenuation coefficient $\mu$ is 0, (14) reduces to $(\xi_{j+1} - \xi_j)(\xi_{j+1} - \xi_j)/\sin \alpha_{jm}$. The support used to evaluate each element in the matrix $M$ is the overlapped area of two natural pixels as shown in Fig. 3. If the support from the overlapping areas is totally within the finite circular support $D$, the matrix element is assigned.
the value given in (14). If the area is totally outside the finite support, the matrix element is assigned a value of zero.

If the intersecting area of two natural pixels is on the finite circular support boundary, the value is given by

$$M_{jm}^{\text{finite}} = \int_{\xi_0}^{\xi_1} \int_{\xi_0}^{\xi_1} \rho(x, y) e^{i \mu x} e^{i \nu y} d\xi d\eta$$

where $\rho(x, y)$ is the characteristic function of the circular disk. The challenge in forming the matrix $M$ is to evaluate those intersections on the boundary. To do this, we divide the overlapped diamond-shaped region in Fig. 3 into 400 subdiamonds, and we use (14) for those subdiamonds within the finite support.

C. SVD Reconstruction Algorithm

Suppose for the data $g$, a solution $\rho$ is desired for the equations

$$A_{jm}^{\text{finite}} n = g_{jm}$$

where $g_{jm}$ are the individual modified projection measurements. Assume that the image $\rho$ can be obtained by determining a vector $q$ such that the backprojection [see (8)] of $q$ gives $\rho$:

$$\rho = A_n^* q.$$  (17)

Substituting (17) into (16), we obtain

$$A_{jm}^{\text{finite}} A_n^* q = g_{jm}.$$  (18)

Referring back to (10), one can see that we can write a matrix equation

$$M q = g$$

where $M$ is given in (14) and (15). The solution to (16) reduces to solving (19).

An SVD is used to find the pseudoinverse of $M$ and to evaluate $q$. The matrix $M$ is symmetric, and for $n$ projection samples only $(n^2 + n)/2$ elements need to be computed. The SVD of $M$ is given by

$$M = U \Sigma U^T.$$  (20)

Here $U$ is an orthonormal matrix containing the singular vectors $u_i$ of $M$, and $\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ with singular values: $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$. The least square solution of $M q = g$ is given by

$$q = U \Sigma^{-1} U^T g = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} (u_i^T g) u_i.$$  (21)

In practice, the upper limit of the summation in (21) is determined by regularization strategies, due to the fact that some of the $\sigma_i^2$ are very small or even zero.

III. METHODS AND RESULTS

A. Methods

SVD Reconstruction: The SVD reconstruction algorithm for the reconstruction of constant attenuated projections using a singular valued decomposition of the exponential Radon transform is summarized as follows:

1) Modify the measured projection data $p$ by the boundary function $\beta(\xi, \nu)$ according to (5).

2) Construct the matrix $M$. A circular disk is assumed as a finite support. If the intersection of natural pixels is within the disk, (14) is used to evaluate the matrix entry. Otherwise the subdivision approximation is used.

3) Compute the SVD of matrix $M$ as indicated in (20).

4) Select a number $N$ and evaluate the least squares solution $q$ according to (21).

5) Backproject $q$ to obtain the final image $\rho$ according to (17).

FBP Reconstruction: The algorithm for the reconstruction of constant attenuated projections using a filtered backprojection (FBP) algorithm is summarized as follows:

1) Modify the measured projection data $p$ by the boundary function $\beta$ according to (5).

2) Convolve the projections using [2]

$$c(x) = \frac{1}{2} \left( \sin \frac{\pi x}{\pi x} - \frac{1}{4} \left( \sin \left( \frac{\pi x}{\pi x/2} \right) \right) \right)$$

$$- \frac{\mu^2}{2 \pi} \left( \frac{\sin \mu x}{\mu x} \right) + \frac{\mu^2}{4 \pi^2} \left( \frac{\sin \left( \frac{\mu x}{\mu x/2} \right)}{\mu x/2} \right).$$  (22)

3) Backproject the filtered projections using the exponential backprojector

$$b(x, y) = \int_0^{2\pi} g(-x \sin \theta + y \cos \theta, \theta) e^{-\mu x \cos \theta - \mu y \sin \theta} d\theta.$$  (23)

B. Computer Simulations

A computer simulation was performed to compare the SVD reconstruction using natural pixels with the FBP reconstruction. The simulations were similar to those previously performed to compare FBP with an SVD reconstruction method using a square pixel image model [3]. The results of these simulations are presented in the Appendix.

A $32 \times 32$ matrix was reconstructed from 32 projection angles sampled over 360 degrees. The projections were sampled from analytical integrals of the phantom shown in Fig. 6(a). In the simulations the projection data were simulated assuming that the pixel size was equal to 7.12 mm. The attenuation coefficient was equal to 0.12 cm$^{-1}$, which is approximately the broad beam attenuation coefficient for 140 keV photons in tissue. The diameter of the circular phantom was 21.4 cm (30 pixels). The phantom
had a 1.4 cm (2 pixels) outer ring of intensity 1000 and an inner background intensity of 500. The bright interior disk had a diameter of 7.12 cm (10 pixels) and an intensity of 1500. The dark interior disk had a diameter of 7.12 cm (10 pixels) and an intensity of 0. The projection bin size was equal to the pixel size of 7.12 mm. In our implementation the width of a natural pixel was equal to the projection bin size of 7.12 mm.

For an attenuation coefficient of 0.12 cm⁻¹ the SVD was calculated once on a SPARCcenter 2000 workstation (40 MHz Super SPARC processor). The SVD calculation of a 1024 × 1024 matrix was performed in approximately 1 and 1/2 hours. On an IBM 3090 this can be accomplished in approximately 3 min.

Ideal and noisy projection data were reconstructed. For noisy projections Poisson noise was added to the simulated projections, which had a total count of 750000 summed over the 32 projections. For both ideal and noisy projections, the SVD reconstruction used all 1024 terms of the expansion in (21). No filter was applied to the reconstruction. The FBP reconstruction used the convolver in (22), which is equivalent to a ramp filter [2]. No additional filtering was performed.

Fig. 4 gives a sample of a few of the eigenimages of the singular value decomposition, and Fig. 5 gives the spectra for attenuation coefficients of 0.12 and 0 cm⁻¹. When \( \mu = 0.12 \), all the singular values are nonzero. However, when \( \mu = 0 \), about half of the singular values are zero. This is because there are many redundant measurements in the case of \( \mu = 0 \).

Fig. 6 shows the FBP and SVD reconstructions with and without noise. The SVD reconstruction handles the noise much better than the filtered backprojection algorithm. Profiles in Fig. 7 indicate that both algorithms compensate for the constant attenuation. However, the FBP has significant overshoots and undershoots.

IV. CONCLUSIONS

This paper presents a method of inverting the exponential Radon transform using an SVD expansion based upon an image basis of natural pixels. The SVD method determines the eigenvalues and eigenimages of the operator composed of the backprojection followed by the projection operator of the exponential Radon transform. For the exponential Radon transform, analytical formulas can be derived for the elements of this matrix operator, except for those elements evaluated for areas of overlapping natural pixels that lie on the boundary of the circular support. The reconstructions have a smooth texture and do not seem to be as sensitive to noise as filtered backprojection [1], [2] or SVD expansions that use the conventional square pixel image representation (see Appendix).

APPENDIX

RECONSTRUCTION USING SVD METHODS ASSUMING SQUARE PIXEL IMAGE MODEL

Computer simulations were performed to evaluate the SVD reconstruction of attenuated projections with constant attenuator using a square pixel image model [3]. In this Appendix, three SVD reconstruction methods are described and compared using computer simulations.

Let \( \rho \) be a column vector that represents a two-dimensional image. Let \( g \) be a column vector that represents the projection measurements. The imaging procedure can be modeled as

\[
A \rho = g
\]  

(24)

where \( A \) is the projection matrix operator, whose transpose \( A^T \) is the backprojection matrix operation. Image reconstruction is to recover \( \rho \) from the projection data \( g \).

It is known that for any arbitrary real \( m \times m \) matrix \( A \) of rank \( r \), there exists an \( m \times m \) orthogonal matrix \( U \) and an \( n \times n \) orthogonal matrix \( V \), such that

\[
A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T
\]  

(25)

where \( \sigma_i \) are the singular values of \( A \), and \( u_i \) and \( v_i \) are the corresponding left and right singular vectors of \( A \).
where

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}. \quad (26)$$

$S$ is a diagonal $r \times r$ matrix and $S = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Here, $\sigma_i$ are the singular values of $A$, the $u_i$ is the $i$th column of $U$, and the $v_i$ is the $i$th column of $V$. The next two expressions follow from (25):

$$A^tA = V \Sigma^2 V^t = \sum_{i=1}^{r} \sigma_i^2 u_i v_i^t \quad (27)$$

$$AA^t = U \Sigma^2 U^t = \sum_{i=1}^{r} \sigma_i^2 u_i u_i^t. \quad (28)$$

The following three reconstruction algorithms follow from (25), (27), and (28), respectively.

(i) Algorithm 1:

$$\rho = \sum_{i=1}^{r} \frac{1}{\sigma_i} v_i [u_i g]. \quad (29)$$

(ii) Algorithm 2:

$$\rho = \sum_{i=1}^{r} \frac{1}{\sigma_i^2} [v_i (A^t g)]. \quad (30)$$

(iii) Algorithm 3:

$$\rho = A^t \sum_{i=1}^{r} \frac{1}{\sigma_i} u_i [u_i^t g]. \quad (31)$$

The variable $r_n$ is an integer between 1 and $r$ (the rank of the matrix), and is chosen to suppress the noise amplification of high frequency spectral terms. Algorithms 1, 2, and 3 compute the SVD in a hybrid of image and projection spaces, in the image space, and in the projection space, respectively. Algorithm 1 does not require a backprojection operation, while Algorithms 2 and 3 require one backprojection operation $A^t$.

A $32 \times 32$ matrix was reconstructed from 32 projection angles sampled over 360 degrees. The phantom and projections were generated as described in the Methods and Results section. In all SVD reconstructions the projection matrix operator $A$ and the backprojection matrix operator $A^t$ were generated using line integrals of the attenuated Radon transform across square pixels to form the weights in the projection and backprojection matrix operators [71]. Algorithms 1 and 2 used 500 singular values and Algorithm 3 used 700 singular values in the reconstructions.

Fig. 8 shows the SVD reconstructions for the three algorithms with and without noise. Algorithm 3 appears to
give the best rests. Even though mathematically the three algorithms should give equivalent results, it is hypothesized that they are not the same because of differences in numerical accuracy between the three algorithms.

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REFERENCES


