A Backprojection Filtering Algorithm for a Spatially Varying Focal Length Collimator

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Abstract—Fan-beam collimators are used in single photon emission computed tomography to improve the sensitivity for imaging of small organs. The disadvantage of fan-beam collimation is the truncation of projection data surrounding the organ of interest or, in those cases of imaging large patients, of the organ itself producing reconstruction artifacts. A spatially varying focal length fan-beam collimator has been proposed to eliminate the truncation problem and to maintain good sensitivity for the organ of interest. The collimator is constructed so that the focal lengths of the holes vary across the face of the collimator with the shortest focal lengths at the center and the longer focal lengths at the periphery of the collimator. The variation of the focal length can have various functional forms but in our work it is assumed to increase monotonically toward the edge of the collimator. A backprojection filtering reconstruction algorithm is derived for this type of collimation. The algorithm first backprojects the projections, then performs a two-dimensional filtering. The algorithm is efficient and has been tested via computer simulations.

I. INTRODUCTION

In single photon emission computed tomography (SPECT), a fan-beam collimator is used with a gamma camera to improve sensitivity for imaging small organs like the brain and heart [1], [2]. The magnification of the fan-beam geometry can place the radiopharmaceutical distribution in some of the tissue outside of the field-of-view. In most cases this tissue does not contain the region of interest but does present a low background which is truncated when projected. The application of the reconstruction filter to these truncated projections produces spikes at the truncation edge in the filtered projections, resulting in ring artifacts in the reconstruction [3], [4].

To resolve the truncation problem, it has been proposed to use a spatially varying focal length fan-beam collimator [5], [6]. The focal lengths increase from a minimum at the center to a maximum at the periphery (edge) of the collimator. Thus, the projection rays converge to focal points of focal lengths that increase as the position of the projection ray moves away from the center of the collimator. The central region of interest is imaged with short focal lengths; whereas tissues at the edge of the body are imaged with nearly parallel rays. The central region of interest is imaged with good sensitivity while at the same time the truncation problem is greatly reduced at the edge of the patient. The spatial variation of the focal length can have various functional forms depending upon the application.

As in all tomographic reconstruction problems, it is desirable for computational efficiency to use a convolution backprojection reconstruction algorithm similar to those that have been developed for parallel [7] and various fan-beam geometries [8]-[11]. Unfortunately, no convolution backprojection algorithm has been derived for the spatially varying focal length fan-beam geometry. Cao and Tsui implemented the inversion formula directly; the formula did not induce a convolution backprojection algorithm [21]. We were able only to develop a summed convolution backprojection algorithm [12] to reconstruct the image, and we conjectured that a convolution backprojection algorithm does not exist for this geometry. The summed convolution backprojection algorithm involves an infinite series of convolutions followed by one backprojection. We found that its finite term implementation may cause artifacts if the focal length function has singular points. Another approach is to use iterative reconstruction algorithms [13] to reconstruct images for this varying focal length geometry. Still another approach would be to use a rebinning method [14] to convert the spatially varying focal length fan-beam projection data into parallel-beam projection data, and then use the parallel-beam convolution backprojection algorithm [7] to reconstruct the image. However, the rebinning step introduces interpolation errors. When the rebinned data are backprojected into the image space another interpolation step is required. Therefore, the rebinning method has two interpolation steps, while a direct reconstruction method has only one interpolation step.

In this paper we develop a direct reconstruction method without rebinning. We derive a backprojection filtering algorithm, which backprojects the projections and then filters two-dimensionally as has been done for standard fan-beam geometries [15]. This algorithm is efficient, yet less well-known than the convolution backprojection algorithm. For the parallel-beam geometry, it has been shown that both the convolution backprojection algorithm and the backprojection filtering algorithm produce nearly the same qualitative results and approximately equal noise sensitivity [16].

II. THEORY

In this paper, we only consider focal length functions \( D(s) \) which are symmetric, i.e. \( D(s) = D(-s) \). We also assume that \( D(s) \) is nondecreasing as \( |s| \) increases and no focal point is within the object being imaged.
A. Spatially Varying Focal Length Fan-Beam Geometry

The geometry for a spatially varying focal length fan-beam collimator is shown in Fig. 1. The focal length varies as a function \( D(s) \), where \( s \) is the projection point on the detection plane measured from the central axis. For tomographic application, the center of rotation is a distance \( R \) away from the image plane.

B. Fan-Beam Projections and Line Integrals

For the flat-detector fan-beam geometry, the projection is given by

\[
p_1(s, \beta) = \frac{D}{\sqrt{D^2 + s^2}} \times \int \int f(x, y) \delta(x \cos \theta + y \sin \theta - t) \, dx \, dy
\]

where

\[
t = \frac{s(D - R)}{\sqrt{D^2 + s^2}}
\]

\[
\theta = \beta + \tan^{-1} \left( \frac{s}{D} \right)
\]

\( D \) is the fan-beam focal length, and \( s \) is the detection point on the detector as shown in Fig. 2. The scaling factor \( D/\sqrt{D^2 + s^2} \) is the sensitivity of a flat fan-beam detector, and was presented in [15]. We define

\[
p_\beta(s) = \frac{\sqrt{D^2 + s^2}}{D} p_1(s, \beta)
\]

\[
= \int \int f(x, y) \delta(x \cos \theta + y \sin \theta - t) \, dx \, dy
\]

C. Point Response Function of the Projection-Backprojection Operator

Considering a varying focal-length geometry as shown in Fig. 1 and using (8), the line-integral of a point source at \((x_0, y_0)\), i.e. \( f(x, y) = \delta(x - x_0) \delta(y - y_0) \), can be written as

\[
p_\beta(s) = \delta(x_0 \cos \theta + y_0 \sin \theta - t)
\]

where \( t \) and \( \theta \) are defined in (6) and (7). If we backproject \( p_\beta(s) \) into the image space, we have the point response function of the projection-backprojection operator:

\[
b(x, y; x_0, y_0) = \frac{1}{2} \int_0^{2\pi} p_\beta(\hat{s}) \, d\beta
\]

where \( \hat{s} \) can be solved by (see Fig. 2)

\[
\hat{s} = \frac{D(\hat{s}) \sqrt{x^2 + y^2 \sin(\beta + \alpha)}}{D(\hat{s}) - R - \sqrt{x^2 + y^2 \cos(\beta + \alpha)}}
\]

and

\[
\alpha = \tan^{-1}(x/y).
\]

If \( D(s) \) is nondecreasing as \( |s| \) increases and the shortest focal point is outside the object, then (11) gives a unique solution of \( \hat{s} \) as illustrated in Fig. 3. If the focal lengths are not long enough, two rays may intersect within the object and not every point \((x, y)\) in the object has a unique solution of \( \hat{s} \). If the...
Fig. 2. The detector is rotated by an angle $\beta$. The fan-beam variables $s$ and $\beta$ are transformed into parallel-beam variables $t$ and $\theta$ using (6) and (7). We assume that there is a one-to-one correspondence between a fan-beam ray and a parallel-beam ray.

shortest focal point is outside the object, and for each point $(x, y)$ in the object there is one and only one solution of $\hat{s}$.

Using (6) and (7), we can evaluate $\hat{t}$ and $\hat{\theta}$ as:

\[
\hat{t} = \hat{s} - \frac{D(\hat{s}) - R}{\sqrt{D(\hat{s})^2 + \hat{s}^2}}
\]

(12)

\[
\hat{\theta} = \beta + \tan^{-1}\left(\frac{\hat{s}}{D(\hat{s})}\right).
\]

(13)

Due to the fact that $D(s)$ is an even function and using (12) and (13), we have

\[
p_{\beta_2}(\hat{s}_2) = p_{\beta_1}(\hat{s}_1)
\]

(15)

where

\[
\hat{s}_2 = -\hat{s}_1
\]

(16)

We notice that each line-integral is measured twice when the detector rotates $360^\circ$. According to (9), we have the same line-integral if we change $\hat{t}$ to $-\hat{t}$ and change $\hat{\theta}$ to $\hat{\theta} + \pi$. Let us denote

\[
\hat{\theta}_1 = \hat{\theta} = \beta_1 + \tan^{-1}\left(\frac{\hat{s}_1}{D(\hat{s}_1)}\right),
\]

\[
\hat{\theta}_2 = -\hat{\theta} = \beta_2 + \tan^{-1}\left(\frac{\hat{s}_2}{D(\hat{s}_2)}\right).
\]

(14)

Therefore, we have

\[
d\beta_1 + d\beta_2 = d\hat{\theta}_1 + d\hat{\theta}_2 = 2d\hat{\theta}.
\]

(18)

We now rewrite (10) as follows:

\[
b(x, y; x_0, y_0) = \frac{1}{4} \int_0^{2\pi} p_{\beta_1}(\hat{s}_1) d\beta_1 + \frac{1}{4} \int_0^{2\pi} p_{\beta_2}(\hat{s}_2) d\beta_2
\]

\[
= \frac{1}{4} \int_0^{2\pi} p_{\hat{s}_1}(\hat{s}_1) (d\beta_1 + d\beta_2)
\]
This equation is exactly the point response of the parallel-beam projection-backprojection operator [7, 17]. Therefore, we have
\[ b(x, y; x_0, y_0) = \frac{1}{r} \]  
(23)

where \( r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \) is the distance from \((x, y)\) to \((x_0, y_0)\).

D. The Reconstruction Algorithm

For the varying focal length fan-beam geometry, the true image \( f \) and the backprojected image \( b \) are related by:
\[ b = f * \frac{1}{r} \]  
(24)

where \( 1/r \) is the two-dimensional point response function of the backprojection operator and \( r \) is the distance from the point source to the reconstruction point. The image \( f \) can be reconstructed by deconvolving the backprojected image \( b \) with the two-dimensional point response function \( 1/r \). Usually, this is done in the frequency domain, where the Fourier transform of \( f \) is the Fourier transform of \( b \) multiplied by a two-dimensional ramp filter. The same reconstruction algorithm can be directly applied to our varying focal length fan-beam geometry.

The reconstruction algorithm consists of two steps:

\textbf{Step 1 Backprojecting:} A pixel-driven backprojector is used to backproject the measured data into the image space. At a certain view angle, a pixel-driven backprojector computes the projection location \( \hat{s} \) on the detector for each pixel \((x, y)\) according to (11). Usually, \( \hat{s} \) is not exactly the sampling point, and its two neighboring samples are linearly interpolated to estimate the projection value at \( \hat{s} \).

If the image fits in an \( n \times n \) array, the backprojection array is \( 2n \times 2n \) which is four times as large as the image array that is returned to the user by the reconstruction algorithm. This is necessary in order to minimize the error due to the convolution result of one period overlapping the convolution result of succeeding period when implementing the discrete Fourier transform.

\textbf{Step 2 Filtering:} We take the two-dimensional Fourier transform to the backprojected image, yielding \( B(u, v) \) where \( u \) and \( v \) are the frequencies with respect to \( x \) and \( y \), respectively. We then multiply \( B(u, v) \) by a two-dimensional ramp filter \( \sqrt{u^2 + v^2} \) with a circular rectangular window function. The final image is obtained by taking the two-dimensional inverse Fourier transform of the filtered result \( B(u, v) \sqrt{u^2 + v^2} \).

E. Pixel-Driven Backprojector

We use a pixel-driven backprojector to backproject the measured data. At each view angle, we need to compute the projection location \( s \) for each pixel \((x, y)\), by solving (11). In our two examples, we have closed-form solutions for \( s \).

Let \( Z = \sqrt{x^2 + y^2} \sin(\beta + \alpha) \) and \( W = R + \sqrt{x^2 + y^2} \cos(\beta + \alpha) \), then (11) becomes
\[ s = \frac{D(s)}{Z} \frac{D(s) - W}{Z} \]  
(25)
In general, (11) may not have a closed-form solution for \( s \) and may have to be solved numerically. However, we do not recommend a numerical solution, because this will make the algorithm to be less efficient.

**F. Ray-Driven Backprojector**

If a more complicated focal length function \( D(s) \) is required, one should consider using a ray-driven backprojector which is easy to implement and does not require to solve (11). However, for the ray-driven backprojector, the number of rays passing through a pixel varies from pixel to pixel. Thus, one must derive a weighting function in the backprojector for normalization, so that the point spread function of the projection-backprojection operator is \( 1/r \). For the fixed focal length fan-beam geometry with a flat detector, this weighting function is basically the reciprocal of the distance from the pixel to the focal point. In general, this backprojection weighting function depends upon the collimator geometry.

A pixel-driven backprojector is widely used in filtered backprojection and backprojection filtering algorithms. A pixel-driven backprojection operator which backprojects the one-dimensional data \( p_\beta(s) \) into a two-dimensional image \( b(i,j) \) is defined by

\[
b(i,j) = \sum_\beta p_\beta(s)
\]

where \( p_\beta \) is the projection angle of the one-dimensional projection data \( p_\beta(s) \) and \( b(i,j) \) is the backprojected value at pixel location \((i,j)\). Usually, \( i \) and \( j \) are integers. The index \( s \) in (35) depends upon the pixel location \((i,j)\), the detector angle \( \beta \), and imaging geometry (e.g., parallel or fan-beam) as well. The calculated value of \( t \) usually is not an integer, while the projection data \( p_\beta(s) \) are sampled at integer values of \( s \). In the computer implementation, \( p_\beta(s) \) is approximated by linear interpolation of its neighboring values.

This section modifies the ray-driven backprojector so that it can replace the pixel-driven backprojector. Assume that at a particular view angle \( \beta \), pixel \((i,j)\) sees \( m+1 \) projection bins: \( k, k+1, \ldots, k+m \) as shown in Fig. 4, where both \( k \) and \( m \) are functions of \( \beta, i, \) and \( j \) according to the imaging geometry. We also assume that the activity is a constant within each pixel. This assumption is commonly made in iterative algorithms. Usually the value of \( p_\beta(s) \) in (35) is approximated...
by the linear interpolation of its two neighboring values. It is proposed here that the value of \( p_\beta(s) \) is approximated by a weighted sum of \( p_\beta(k), p_\beta(k + 1), \ldots, p_\beta(k + m) \), and their weighting factors are proportional to their line-length within the pixel \((i, j)\). That is,

\[
p_\beta(s) = \frac{L_k p_\beta(k) + L_{k+1} p_\beta(k+1) + \cdots + L_{k+m} p_\beta(k+m)}{L_k + L_{k+1} + \cdots + L_{k+m}}
\]

(36)

where \( L_{k+l} \) is the line-length of the ray for bin \( k + l \) within pixel \((i, j)\) and \( p_\beta(k + l) \) is the measured projection at bin \( k + l \). The sum of the weighting factors for each pixel at each projection view angle is unity. In the computer code, initially \( p_\beta(s) \) and \( w(i, j) \) are set to zero and (36) is implemented recursively as:

\[
p_\beta(s) \leftarrow \frac{w(i, j)p_\beta(k) + Lp_\beta(k + l)}{w(i, j) + L}
\]

(37)

\[
w(i, j) \leftarrow w(i, j) + L
\]

(38)

where \( i \) and \( j \) are indices of pixel \((i, j)\), \( k + l \) is the index of current detection bin, and \( L \) is the current line-length within the pixel. Therefore, the pixel-driven backprojector (35) can be rewritten in the ray-driven version as follows:

\[
b(i, j) = \sum_{\beta} \frac{\sum_{k=1}^{M} L_k(b(i, j)p_\beta(k))}{\sum_{k=1}^{M} L_k(b(i, j))}
\]

(39)

where \( L_k(b(i, j)) \) is the line-length within the pixel \((i, j)\) of the ray for bin \( k \). Equation (39) assumes that the detector consists of \( M \) bins.

III. COMPUTER SIMULATIONS

A two-dimensional Shepp-Logan head phantom [18] was used in our computer simulations. The computer generated phantom is shown in Fig. 5. The number of projection angles was 128 over 360°. The distance \( R \) from the center of rotation to the detector was 64 pixels (12.5 cm). The detector size was 160 bins (31.25 cm). Images were reconstructed in 128 by 128 pixel (25 cm by 25 cm) arrays. However, the backprojected images were in 256 by 256 arrays. We used larger backprojected arrays in order to reduce the wrap around caused by frequency domain filtering. Three different focal length functions \( D(s) \) were used in computer simulations.

A. Using Pixel-Driven Backprojector: Two of the three focal length functions were exactly the same as those in [12], so that the results could be compared. For the first focal length function \( D(s) = a + ks^2 \), we used \( k = 0.03 \) and the minimum focal length \( a = 170 \) pixels (33.2 cm). For the second focal length function \( D(s) = a + k|s| \), we used \( k = 0.8 \) and the minimum focal length \( a = 250 \) pixels (48.83 cm). These parameters were chosen so that the projection data were not truncated. The reconstruction algorithm is given in Section 2.D. Since we used two different focal length functions, we used two different phantom generating routines and pixel-driven backprojectors.

B. Using Ray-Driven Backprojector: Finally, the modified ray-driven backprojector proposed in Section 2.F was used for the focal length function

\[
D(s) = a + ks^4
\]

(40)

where \( a = 170 \) pixels, and \( k = 0.000004 \). That is, the shortest focal length was 170 pixels (33.2 cm), and the longest was 335 pixels. There is no easy way to implement a pixel-driven backprojector for this imaging geometry, because the value of \( s \) cannot be evaluated analytically. However, it is straightforward to implement a ray-driven backprojector using (39).

The algorithm described in Sections 2.D and 2.F, including the projection data generating routine, was coded in Fortran 77 by modifying the Donner Laboratory [19] on a SUN SPARC-II computer. The total data generating and image reconstruction time was 4 minutes for the first focal length function, 2 minutes for the second focal length function, and 20 seconds for the third focal length function with ray-driven backprojector, respectively. The differences in computation times are mainly due to the complexity of the backprojectors in the calculation of \( s \). Fig. 6 shows the reconstructed images with the pixel-driven backprojectors, and Fig. 7 shows the reconstruction with the ray-driven backprojector. All images were reconstructed in a circular region, and outside the circular region the pixel values are zero. All the reconstructed images are fairly accurate, however, their background noise textures are different and depend upon the collimation functions \( D(s) \). We did not attempt to determine an optimal collimation function in this paper.

In our computer simulations, the projections were line integrals of the Shepp-Logan phantom. For the experimental data \( p_\beta(s, \beta) \), one had to convert the measured data into line integrals \( p_\beta(s) \) according to (8) before reconstructing the image.

IV. DISCUSSION

Collimators with a spatially varying focal point have been proposed in SPECT to improve the sensitivity over that of parallel-hole collimators for imaging small organs and to reduce the truncation problem suffered by higher sensitivity fan-beam collimators, especially when applied to image the
heart. For this particular geometry, we were not able to derive a conventional convolution backprojection algorithm. We could only arrive at a reconstruction algorithm in the form of a backprojection of an infinite series of convolutions [12]. However, it was demonstrated that for the spatially varying focal length function \( D(s) \), for example, \( D(s) = a + k|s| \), which has a singularity at \( s = 0 \), an artifact, seen as a dark hole, appears at the center of rotation. Whereas in this paper, we show that the backprojection filtering algorithm does not give this artifact.

In situations where the one-to-one condition is not met such as the case where there are \( M \) fan-beam rays corresponding to one parallel ray, we can prescale each of those fan-beam rays by \( 1/M \) and use the backprojection filtering algorithm to reconstruct the image. As a special example, \( D(s) = |s| \) is equivalent to using two camera heads connected at a right angle [20].

In this paper, we require that \( D(s) \) be symmetric in order to obtain the shift-invariant point response function \( 1/r \). Roughly speaking, the focal length function \( D(|s|) \) is arbitrary as long as it gives a one-to-one correspondence between a fan-beam ray and a parallel-beam ray, considering the trajectory (orbit) of the detector rotation. If the shortest focal length \( a \) covers the field-of-view as shown in Fig. 1, then the one-to-one condition is satisfied for our three examples: \( D(s) = a + ks^2 \), \( D(s) = a + k|s| \), and \( D(s) = a + ks^4 \).

Even if convolution backprojection algorithms do not exist, it is more often than not that backprojection filtering algorithms do apply for several of these geometries. In this paper, we have shown that for variable focal length fan-beam geometries a backprojection filtering algorithm also applies. This algorithm is more accurate and more efficient than the summed convolution backprojection algorithm [12] which we developed earlier.

From our computer simulations, the backprojection filtering algorithm produces almost exact reconstructions without any visible artifacts. This agrees with the results presented in [16]. However, the backprojection filtering algorithm may cause a small dc-offset, i.e. the reconstruction is shifted by a small constant value. We did a phantom study with a fixed focal length fan-beam geometry with a focal length of 500 pixels (see Fig. 8). Like the studies with varying focal length fan-beam geometries, this study also showed a small dc-shift—about 6.7%. This phenomenon is caused by the discretized two-dimensional ramp-filter \( \sqrt{u^2 + v^2} \), which is zero at the zero frequency. The dc component in the backprojected image is thus lost. Another cause is that a finite array is used to store the backprojected image which does not have a finite support. One remedy is to estimate the dc component from the projection data and to add the dc component back to the reconstructed image. Another remedy is to use spatial domain two-dimensional convolution to replace the frequency domain ramp filtering. However, in this paper we do not attempt to solve this general dc-shift problem in a backprojection filtering algorithm.
backprojector in order to obtain the radius function \( c(\beta) \). If the detector rotates in a variable orbit with a weighting factor is required in the backprojector in order to obtain the \( 1/r \) point spread function. This result will be published in a separate paper.

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**REFERENCES**


