Numerical Study on the Quasi-periodic Behavior in Coupled MEMS Resonators

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Abstract Phenomena such as quasiperiodic oscillations and synchronization can occur in coupled resonators. In the design of the coupled system, it is necessary to understand the specific properties of the individual resonator and the coupled system. In this paper we focus the numerical simulation of coupled MEMS resonators. We discuss the collective behavior of the coupled system according to the change in the coupling parameter.

Key words Nonlinear dynamics, Coupled MEMS Resonators, Quasiperiodicity

1. Introduction

As the MEMS technology matures, MEMS resonators are being substituted for crystal oscillators in automobiles (e.g. engine, drive train control systems, and various sensors) and wireless communications (e.g. reference clocks in kHz range, filters, and resonators in MHz to GHz range) [1][2]. MEMS devices can exhibit complex dynamics such as spring-hardening, spring-softening, parametric resonance, and hysteresis [3]. Therefore it is important to study the dynamics underlying the MEMS resonators. Complex dynamics of the coupled oscillators have already been reported in the literature [4-6]. In this research we focus on the MEMS resonators fabricated in MEMS-only process such as SOIMUMPs; how they can be coupled by discrete electronics, and what sort of properties or behavior they can exhibit when they are coupled in a unidirectional coupling scheme. In this paper a numerical study of laterally driven comb-drive resonators is shown. First, the design process of a single comb-drive resonator is explained. Next, the coupled system consisting of three resonators is discussed.

2. A Single Resonator

Figure 1 shows a simple resonator fabricated in SOIMUMPs process. Here the resonator is a perforated mass suspended by folded springs. The springs are attached to a truss. The device is designed to be symmetric about x and y axes to provide stable oscillations. By applying ac voltage between Input pad and Anchor pad, the comb fingers generate force between them such that the mass vibrates in x-direction. It is anticipated that the vibration in this type of resonator is only in x-direction due to the folded springs and the thickness of the structure. These factors reduce out-of-plane vibrations and axial loading which are often the causes of unstable and unnecessary vibration in the other axes.

SOIMUMPs is a simple Silicon-on-insulator (SOI) patterning and etching process offered by MEMSCAP, Inc. Fig. 2 shows the layers used in SOIMUMPs process. The end result is 25 μm thick doped Silicon as the structure layer followed by 2 μm thick Oxide layer patterned and etched on 400 μm thick Substrate. The substrate underneath the structure is etched away and completely removed. Top Metal layer is used for bond-pads.
The following equation represents the MEMS resonator with nonlinearity,

\[ m \ddot{x} + c \dot{x} + k_1 x + k_3 x^3 = A_d \cos(\omega_d t) \quad (1) \]

Where \( m \) denotes mass, \( c \) is the damping constant to account for various mechanical and sense electronics losses, \( k_1 \) and \( k_3 \) correspond to the linear and nonlinear mechanical spring constants respectively. \( \omega_d \) is the excitation frequency and \( A_d \) is the excitation amplitude. Nonlinearity in MEMS resonators appears due to the material anisotropicity (e.g., nonuniform material), large vibrations, variation in individual elements, and/or a combination of all of the above. Nonlinear behavior such as spring-hardening effect in fixed-fixed beam resonator and comb-drive resonator has been reported in MEMS research [7-9]. Normally the nonlinearity is characterized so that the device can be designed and operated in linear region. However in this work nonlinear behavior is emphasized as it can be useful in a coupled system. The following equation represents the MEMS resonator with nonlinearity in dimensionless form by substituting \( \tau = \omega_0 t \) in Eq. (1).

\[ x'' + \alpha x' + x + \beta x^3 = \epsilon \cos(\omega \tau) \quad (2) \]

Where (') and ("') represent first and second order derivatives with respect to the normalized time \( \tau \). \( \omega_0 \) denotes natural frequency used in normalization. Then it follows that \( \alpha = c/(m \omega_0^2) \), \( \beta = k_3/(m \omega_0^2) \), \( \epsilon = A_d/(m \omega_0^2) \), and \( \omega = \omega_d/\omega_0 \).

### 2.1 Parameter Values

The value of the mass \( m \) is the least affected parameter during MEMS fabrication [10]. This value was calculated from \( m = Ah \rho = 5.92 \times 10^{-9} \text{kg} \); where \( A \) is the total area calculated from the layout, \( h \) is the structural thickness, and \( \rho \) is the density of Silicon (about 2.33x10^3 kg/m^3). From the spring design equations for folded fixed-guided beams, the spring constant \( k_1 \) was obtained as 30.97 N/m. Using these values the natural frequency was approximated as 11512 Hz from \( f_0 = \omega_0/2\pi \); where \( \omega_0 = \sqrt{k_1/m} \). The value of \( k_1 \) was verified by Finite Element Analysis (FEA) software by plotting the curve of applied force vs displacement and by calculating the slope of this curve. The value of \( f_0 \) was also verified by running eigen-mode analysis and frequency sweep analysis in FEA software. The resonator was fabricated in SOIMUMPs process in which the substrate under the spring-mass structure is completely removed. Therefore it is anticipated that the loss of energy and lowering of the Q factor will be due to various parasitic capacitances of the pads and read-out electronics. Additionally the energy loss can also be due to the structure instability during oscillations, material damping itself, and a finite volume of the anchor. Moreover damping is even higher at atmospheric pressure (about 101 kPa). However for the simulation purposes, \( Q \) of the resonator was taken as 282 which is on par with the value observed in the experiments. From \( Q = \sqrt{k_1 m/c} \), the damping constant \( c \) was estimated 1.5x10^6 Ns/m. For the simulation purposes, \( k_3 \) was taken as 100. Using the above values, \( \alpha \) was calculated as 0.0035, and \( \beta \) was calculated as 3.23 in Eq. (2). After fixing all other parameters, the tunable parameters in the dimensionless form are \( \epsilon \) and \( \omega \). Fig. 3 shows the amplitude and phase response curves for \( \epsilon = 0.001 \) while sweeping \( \omega \).
As expected, linear response appears when \( \beta = 0 \). After changing \( \beta \) to 3.23, the resonator shows spring-hardening response and hysteresis while sweeping the frequency up and down. The single resonator has a uniwell potential function. By applying a sinusoidal external force we can observe a change in the equilibrium point \( x = 0 \) as it becomes unstable and a stable periodic cycle with same frequency as the excitation frequency appears. As the excitation frequency is increased another stable periodic cycle and an unstable saddle cycle appear via saddle-node bifurcation as shown in Fig. 3a. The stability was verified by the corresponding eigenvalues. Further increasing the frequency of the excitation signal destroys the saddle cycle and one periodic cycle remains. The width of the hysteresis region, in which the jump phenomenon occurs, depends on excitation signal amplitude and damping constant. For an example, with \( \epsilon = 0.01 \) the amplitude and the width of the hysteresis region significantly increase as observed in the simulation.

3. Unidirectionally Coupled MEMS Resonators

3.1 Coupling Topology

When two nonlinear MEMS resonators are coupled together and are excited by an external force they exhibit in-phase oscillations if the coupling is strong and out-of-phase oscillations if the coupling is weak [11]. In-phase oscillations are characterized by phase difference of zero and out-of-phase oscillations are characterized by phase difference of \( \pi \). This behavior can be further exploited such that weakly coupled MEMS resonators in a network can be used for pattern recognition [12]. Given symmetric coupling, the network can converge to a phase-locked pattern such that their frequencies would be equal but their phases need not be equal [12]. The concept of phase-locking can be extended to a ring of coupled resonators. Furthermore, two types of coupling topologies are possible: a) diffusive and b) direct. Diffusive coupling is defined by coupling related to the difference between the displacement variables. Direct coupling is defined when only the neighboring element is considered as shown in Fig. 4.

Furthermore for the coupling to be tunable, symmetrical, and electrical, direct coupling topology is an appropriate choice. The dynamics of a single resonator in the coupled system with direct-coupling from the neighboring element can be described as,

\[
x''_j + \alpha x''_j + x_j + \beta x^3_j = \epsilon \cos(\omega \tau) + \lambda_{j-1} x_{j-1}; \quad j = 1, 2, 3
\]

Where \( x_j \) denotes displacement variable of a single resonator in the coupled system, and \( \lambda_{j-1} \) is the coupling strength. Here subscript \( j-1 \) denotes the previous element as shown in Fig. 4. To reduce the complexity and the number of parameters of a coupled system, it is assumed that a) the coupling is linear and identical (i.e. \( \lambda_1 = \lambda_2 = \lambda_3 \)), b) the number of elements is 3, and c) all other parameters are identical. The parameter space now contains three tunable variables: coupling strength \( \lambda \), excitation force amplitude \( \epsilon \) and excitation force frequency \( \omega \).

3.2 No Excitation Force

Figure 5a shows one parameter bifurcation diagram for \( x_1 \)

![Fig. 4 Unidirectional Coupling Scheme](image)

![Fig. 5 Simulation of a single resonator within the coupled system: (a) Bifurcation diagram with \( x_1 \) and \( \lambda \), (b) time-series for \( \lambda \) close to the Hopf bifurcation point](image)
as $\lambda$ is varied. As shown in Fig. 5a, while sweeping the coupling strength $\lambda$ (from right to left) the trivial equilibrium point $x_1=x_2=x_3=0$ loses its stability by going through supercritical Hopf Bifurcation (HB) at a critical value of $\lambda$. Periodic rotating waves appear as shown in Fig. 5b. Fig. 5b shows the time-series of the three resonators when $\lambda=-4.044\times10^{-3}$ right after the bifurcation point. Note that in this simulation nonidentical initial conditions were used. Nonidentical initial conditions always occur in actual device given the noise in the electronics and the process variation. At the critical value of $\lambda$, where the oscillations begin, the frequency appears close to 1. Fig. 6a shows the time-series of the coupled system at $\lambda=-4.08\times10^{-3}$. The power spectrum shown in Fig. 6b confirms the single oscillation frequency. As seen from Fig. 6c, the phase-space contains a circle. Furthermore, all Lyapunov exponents of the system stay negative after approaching steady state which indicates stable periodic cycle.

As $\lambda$ is further changed, frequency of the oscillations increases. These periodic cycles are stable up to a certain $\lambda$, after which the system goes through a secondary Hopf Bifurcation. Fig. 7 shows the response of the system when $\lambda$ is changed to $-4.28\times10^{-3}$. As shown in Fig. 7a, the original phase-locked state between the three oscillators is now destroyed and the system shows amplitude-modulated periodic cycles. There are frequencies other than the original oscillatory frequency present as seen from the power spectrum in Fig. 7b. Note that the third harmonic components begin to rise for this value of $\lambda$. From a circular phase-space in Fig. 6c, phase-space now evolves into torus as seen in Fig. 7c. While checking steady-state condition of the Lyapunov exponents, two of them asymptotically approach zero confirming the quasiperiodicity. Furthermore Poincare map generated for this value of $\lambda$ clearly showed a closed loop. The transition from equilibrium point (E) to periodic cycles (P) and onto the quasiperiodic cycles (Q) and the corresponding boundaries are marked in Fig. 5a. These values of the coupling parameter and the corresponding regions were also verified in Auto [14]. A similar behavior was observed for a positive range of $\lambda$ with critical value of $4.04\times10^{-3}$.

### 3.3 With the Excitation Force

With nonzero $\epsilon$, the state of the coupled system is changed dramatically. Here it is assumed that same external force is applied to all three resonators. Figs 8, 9 and 10 show the three states for $\lambda=-4.08\times10^{-3}$ with $\epsilon=0.001$, $\epsilon=0.01$, and $\epsilon=0.1$. For these simulations $\omega$ was set to 1.05. Here we see the coupled system in transition from Fig. 8 to Fig. 10. For $\epsilon=0.001$ (in the order of $\lambda$), the system begins to lose its original synchronized state (see Fig. 6a) and the interaction of the three oscillators results in the rise of the third harmonic components as seen in Fig. 8b. By changing $\epsilon$ to 0.01, the system becomes more unstable as seen in Fig. 9. As the system is driven even harder by increasing $\epsilon$ to 0.1, it is fully frequency-locked and phase-locked to the excitation signal. Note the clear appearance of the third harmonic at $\omega=3.15$ in Fig. 10b. In the fully synchronized state, the phase-space become linear as seen in Fig. 10c. For lambda $=4.28\times10^{-3}$, the coupled system shows many spectrum components until $\epsilon$ is sufficiently high at which full entrainment, similar to Fig. 10, occurs.
Fig. 8 Simulation of coupled system for $\lambda=-0.00408$ and $\epsilon=0.001$
(a) Time-series for $x_1, x_2, x_3$, (b) Power spectrum for $x_1$ and the zoomed view around fundamental frequency and its third harmonic, and (c) Phase-space for $x_1, x_2, x_3$

4. Discussion

Because the coupled system is symmetric (e.g. if the coupling is reversed, the system is invariant in regards to the rotation), the effect of changing the coupling strength results in the solutions that have spatio-temporal symmetry [4]. Group theory combined with symmetry breaking Hopf bifurcation theory describes four possible states of phase-locking behavior a ring of 3 coupled oscillators: a) completely synchronized state, b) one third of a cycle out-of-phase with each other, c) two synchronized to each other with third resonator with unrelated phase, or d) two out of synchrony and the other with twice as much frequency [13]. These states described generic behavior of the coupled system and therefore it may not be possible to observe all four states when the coupling strength $\lambda$ is changed for the coupled system under study. In a ring of $n$-coupled elements (for odd $n$), the system exhibits $2\pi/n$ phase difference between neighboring elements [5] [13]. Similarly for all stable periodic cycles, $2\pi/3$ phase difference appears in this system where $n$ is 3. All periodic solutions of the individual resonator maintain a fixed phase difference with the neighboring elements. It is important to note that when the initial conditions are identical, oscillations die out due to fully synchronized phases and amplitudes. In this case the coupling terms can not maintain full grown oscillations. Another important point is that when the initial conditions are nonidentical and when the coupling strength is changed to a specific value, the symmetry between the three oscillators breaks and the system goes through symmetry breaking bifurcation. At this point, the stable periodic rotating waves loose stability and a torus without the spatio-temporal symmetry appears as seen in Fig. 7c. The presence of the excitation force on the coupled system in a priori oscillatory state, such as the one shown in Fig. 6a, causes the system to break the symmetry once again. With small amplitude of the excitation force and different excitation frequency, the coupled system is destabilized into the torus as seen in Fig. 8c while maintaining its original frequency of oscillations. As seen in Fig. 8b, the spectrum shows a distinct harmonic at the ex-
citation frequency (e.g. $\omega=1.05$). Even though the presence of the excitation force detunes the individual resonators in the coupled nonlinear system, their interaction causes third harmonic components to appear at very low level. As seen in Fig. 9, further increasing the excitation force amplitude creates more detuning in previously partial-synchronized state of the coupled system and we see a complicated torus and many frequency components around the original frequency of the coupled system. Similarly the system shows many frequency components around the third harmonic of the excitation force and the third harmonic of the original frequency of the coupled system. This state shows the effects of the force and detuning of the frequencies; the two factors that govern how the system will be eventually synchronized. At this value of the excitation amplitude, the phase difference between the coupled system and the force can not be fully eliminated. However as seen in Fig. 10, further increasing the excitation amplitude now causes the system to be fully entrained as the phase difference between the excitation force and the coupled system becomes zero. In this study, the individual resonators have the same parameters. However the final state of the coupled system shown in Fig. 10 is similar to the injection-locking of oscillators with different resonance frequencies by an external force [11]. Another useful advantage of the coupled nonlinear system with symmetry is that given specific parameter values it has inherent 1/3rd subharmonic oscillations accompanied with quasiperiodic oscillations as noted in [15]. Then the coupled system can be entrained at exactly 3 times the oscillations frequency in the a priori oscillatory state. Another region of the parameter space, when $\lambda$ is below the critical coupling strength, shows that the system can start oscillations only in the presence of the excitation signal. This and other regions of the parameter space are being analyzed in the current research. These properties make the coupled nonlinear resonators an interesting system. As an ongoing effort, the MEMS devices such as the one shown in Fig. 1, are being characterized individually and in a unidirectional coupling scheme in the experimental study.

5. Summary

In this paper we have discussed the design of a single MEMS resonator and the simulation of a system of coupled MEMS resonators. By numerical simulations, it was shown that the coupled system can oscillate without an excitation signal and also exhibit quasiperiodic cycles by changing the coupling strength. Adding an external force to the coupled system with a priori oscillatory state changes the dynamics significantly. Phase-locked and frequency-locked states of the coupled system were also discussed.

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