RESEARCH STATEMENT

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The notion of the spectrum of a self-adjoint operator has proved to be of great interest and use in various branches of mathematics. It is natural to try and extend the notion to n-tuples of operators. In 1999, Charles Akemann, Joel Anderson and Nik Weaver came up with the notion of a spectral scale [3]. The setting is as follows. Let M be a finite von Neumann algebra equipped with a normal, faithful tracial state, τ. Elements of M can be thought of as bounded operators on some Hilbert Space, H. For a given self-adjoint $b \in M$ the corresponding spectral scale, $B(b) := \{(\tau(a), \tau(ba)) \mid 0 \leq a \leq 1\}$, yields information about the spectrum of b in a nice geometric way. For example, the slopes of the lower boundary curve of $B(b)$ correspond to points in the spectrum of b, while corners on the lower boundary curve correspond to gaps in the spectrum. There is an analogue to this for n-tuples of self-adjoint operators. If $(b_1, ...b_n)$ is an n-tuple of self-adjoint operators in M, then the corresponding spectral scale is given by $B(b_1, ..., b_n) := \{(\tau(a), \tau(b_1 a), \tau(b_2 a), ..., \tau(b_n a)) \mid 0 \leq a \leq 1\}$. Properties of $B(b_1, ..., b_n)$ were discovered by analyzing two dimensional slices of $B(b_1, ..., b_n)$, thus reducing to the single operator case. A key property of $B(b_1, ..., b_n)$ is that it is a compact, convex subset of $\mathbb{R}^{n+1}$.

My dissertation discussed a generalization of this: instead of considering self-adjoint elements of M, we considered self-adjoint elements of M_*. Using the theory of non-commutative integration, and especially a beautiful expository paper by Edward Nelson [4], it can be shown that for each self-adjoint $g \in M_*$, there exists an operator, b, possibly unbounded, affiliated with M, such that $\tau(ba)$ makes sense for every $a \in M$. Hence, a lot of the results obtained by Anderson, Akemann, and Weaver were also valid in this situation. The main difficulties were issues of domain, but fortunately the existence and utility of τ helped take care of those difficulties. These results are now available in a preprint at http://marcus.whitman.edu/∼willsmd/GenResearchInfo/ResearchFiles/ spectralscales i.pdf

Our current research in these matters asks under what circumstances a given spectral scale determines an n-tuple. In the bounded situation, the answer was that if M is commutative, then the spectral scale determines the n-tuple, but in the non-commutative situation, the spectral scale is not enough. As a result, the notion of a ‘complete spectral scale’ was introduced. Roughly, a complete spectral scale is a sequence of matrices with complex entries depending on M, τ, and $(b_1, ..., b_n)$. It turns out that the complete spectral scale determines the n-tuple.

We expect to have to do the same thing in the unbounded situation, but our task is complicated by the fact that the number of conditions we need to check in the unbounded case appears to be uncountable, and do not depend directly on the g_i's. We believe that we can reduce to a countable number of conditions, and those conditions will depend directly on the g_i's.

Separately, we asked the following question: if a given sequence of n-tuples of self-adjoint elements in M converged in some fashion to an n-tuple of self-adjoint elements in M, do the corresponding spectral scales converge in some topology? Since spectral scales are compact subsets of Euclidian space, it seems that the natural topology to work with is the Hausdorff metric on $\mathbb{R}^{n+1}$, which we now define [5].

Let (X, d) be a metric space, with A and B nonempty subsets of X. Define $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$. Set $A_\epsilon = \{x \in X \mid d(x, A) < \epsilon\}$ and $B_\epsilon = \{x \in X \mid d(x, B) < \epsilon\}$. Define $d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset B_\epsilon, B \subset A_\epsilon\}$. Then $d_H(A, B)$ is the **Hausdorff distance** between A and B.

We discovered that if a given sequence of n-tuples of self-adjoint elements in M converges strongly in each coordinate to an n-tuple in M^n, then the corresponding spectral scales do indeed converge in the Hausdorff metric. If we replace 'strongly' in the above statement with 'weakly', the result no longer holds.

We also discovered that if X is a normed space, and A and B are bounded, closed, and convex, then $d_H(A, B) = d_H(\partial A, \partial B)$. This is the main result in my first paper, which I submitted to the *Journal of Convex Analysis* (JCA) on 1st November 2004. The paper was accepted in October 2005.

In addition to answering the questions mentioned above, I also hope to generalize additional results [1],[2] that tie together the notion of the numerical range of a (not necessarily self-adjoint) operator, and its spectral scale. Given a bounded operator, $b = b_1 + ib_2$, where b_1 and b_2 are self-adjoint, we can define the spectral scale of b to be $B(b) := B(b_1, b_2)$ and the numerical range of b to be $W(b) = \{\langle b\xi, \xi\rangle \mid \xi \in H, ||\xi|| = 1\}$. It turns out that the boundary of $W(b)$ is exactly the set of radial complex slopes on $B(b)$ at the origin.

In the unbounded situation, we start with $g = g_1 + ig_2 \in M_*$. We can certainly find b, b_i such that $g(a) = \tau(ba)$ and $g_i(a) = \tau(b_i a)$ for every $a \in M$, but it is not obvious that there is any relationship between b and the b_i's, and hence it is not clear that there is any relationship between $W(b)$ and $B(b_1, b_2)$. We believe that we can resolve this issue.

More generally, I am interested in convexity theory and pure functional analysis, as well as operator algebras. I hope to explore open problems in all of these areas in future work.

References

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