THE MYSTERIOUS CASE OF THE WEAK DEFORMATION RETRACTION

MIKE "A.C. DOYLE" WILLS

Of all the mysteries that Mr. Sherlock Holmes and I investigated, I can safely say that none involved less blood than the affair of the weak deformation retraction.

We were spending a quiet evening in our rooms at (Math) 221B Baker Street, smoking our pipes and reading our post. Outside, there was a thunderstorm complete with freezing rain.

"I say, Holmes. How long do you think this frightful weather will keep up?"

"It is January, my dear Watson. We are in London. The sleet will be with us until March at least, followed by four months of intermittant rain".

"I dare say you're right".

The post consisted mostly of bills, but there was a letter that appeared to be from New York. It was addressed to both of us, so I took the liberty of opening it.

The letter appeared to be from New York's famous Dr. A. H. We had assisted H. previously with our work on the case of the one-point compatification. I read it aloud to Holmes.

"My dear Holmes and Watson,

I trust that you are both in good health. Thank you again for helping me with that knotty compactification mystery.

Another problem has arisen, and I suspect that only you (or possibly Mr. M.W. of Santa Barbara) can help me.

You will doubtless recall the notions of homotopy and deformation retraction. A Dr. J.M. from Texas has introduced me to a slightly different notion- that of a weak deformation retraction. The idea is as follows:

Definition 1. Consider a space X, and a subspace A. Suppose that we have a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$, $F(x, 1) \in A$ and $F(a, t) \in A$ for every $x \in X$, $a \in A$, and $t \in I$. Then F is a **deformation retraction of X to A in a weak sense**.

The question is this:

Question 2. *Suppose we have $i : A \to X$, the inclusion map given by $a \mapsto a$. Given an F as above, can we make any conclusions about i?*

Sin'c'ly
AH "

"Well, Holmes, what do you make of that?"

"I have not come to any conclusions at this time, apart from the fact that H. is a little nervous about this."

"How the deuce did you come to that conclusion?"

"Elementary, my dear Watson. You and I both know Mr. M.W. since we assisted him in the case of the rogue epsilon. He's not from Santa Barbara; he's from Berlin."

"Of course! He was only in Santa Barbara to attend a convention."

"Precisely, Watson. H. is extremely meticulous. He would not have made that slip unless he was gravely distracted."

"By Jove, we must help him!"

"Indeed. Let us think on this for a while."

I sat in silence, listening to the snow hitting the window and the comforting roar of the fire. Holmes stood by the mantle, smoking and thinking. After ten minutes, he stood up straight and declared: "Clearly i is a homotopy equivalence."

"How the d_il did you work that out, my dear Holmes?"

"I simply applied all the pertinent facts and the conclusion dropped out."

"Please explain."

"Do you recall what a homotopy equivalence is?"

"Of course. i is a homotopy equivalence if there is a map $g : X \to A$ such that $ig \simeq \mathbf{1}$ and $gi \simeq \mathbf{1}$.

"Precisely, Watson. The problem is thus a simple matter of finding the correct g."

Thunder rolled. It rolled a six.

"Well, Holmes, the obvious choice would be $g(x) = F(x, 1)$ for every $x \in X$."

"Lucidly put, Watson! Now it is sufficient to check that this g will in fact serve our purposes. To fix our ideas, let $H(x,t) = F(x,t)\mid_{A\times I}$. The claim is that F will be the homotopy between ig and $\mathbf{1}$, while H will be the homotopy between $\mathbf{1}$ and gi. Let us check that:

$$i[g(x)] = i \circ F(x, 1) = F(x, 1).$$

This makes sense, since we know that $F(x, 1) \in A$ for every $x \in X$."

"Indeed, Holmes, all this is very clear. But I'm d_ed if I can work our the other direction."

"My dear Watson, you have all the necessary facts at your disposal. Observe that the range of H is A. Therefore, $H(a, 0) = \mathbf{1}$ is homotopic to $H(a, 1)$. (The continuity is immediate.) But

$$g[i(a)] = g(a) = F(a, 1) = H(a, 1)$$

for every $a \in A$. Thus $gi \backsimeq \mathbf{1}$ and we are done."

"Well done, Holmes. I shall send the solution to H. forthwith."

"Indeed. the New York police do not look kindly on confused topologists. He will be much relieved. More tea?"

I shook my head and said "It's amazing how you so easily solve these mysteries, Holmes!"

"My good Doctor. It is simply a case of observing the facts and drawing the correct conclusions from them. Also, I pray to the powers that be that I do get it right because otherwise I would end up looking pretty silly."

"Perish the thought, my dear Holmes!"

FINIS

Author's Note (to the Professor): I promise not to do this again, but once I saw the connection between our course number and Sherlock Holmes, I felt that it had to be done at least once.

M. *A.C.D.* W.

Department of Mathematics, University of California-Santa Barbara,Santa Barbara, CA, 93106
E-mail address: mikewill@math.ucsb.edu