More Homework for Math 4110

7. **Problem:** Find an example of a noncyclic group, all of whose proper subgroups are cyclic.

   Solution. The group $U(8) = \{1, 3, 5, 7\}$ is noncyclic since $1^1 = 3^2 = 5^2 = 7^2 = 1$ (so there are no generators). The only proper subgroups are $\{1\}$, $\{1, 3\}$, $\{1, 5\}$, and $\{1, 7\}$, which are all obviously cyclic.

14. **Problem:** Suppose a cyclic group G has exactly three subgroups: G (the whole group), $\{e\}$ (the identity subgroup), and a subgroup of order 7. What is $|G|$?

    Solution. If we let $n = |G|$, then G has a unique subgroup of order k for every divisor k of n. The given information about the subgroups implies that the only divisors of n are 1 and 7. So n must be a positive integer that is only divisible by 1 and 7. This forces $n = 49 = 7^2$. (If we replace 7 with a prime p, then the order of G must be p^2.)

24. **Problem:** Suppose G is a group and $a \in G$. Then $\langle a \rangle$ is a subgroup of $C(a)$.

    Solution. It suffices to show that $\langle a \rangle \subseteq C(a)$. If $x \in \langle a \rangle$, then $x = a^k$ for some $k \in \mathbb{Z}$. Note that $xa = a^k a = a^{k+1} = aa^k = ax$, so by definition $x \in C(a)$, as desired.

28. **Problem:** Let a be a group element that has infinite order. Prove that $\langle a^i \rangle = \langle a^j \rangle$ if and only if $i = \pm j$.

    Solution. If $i = j$, it is clear that $\langle a^i \rangle = \langle a^j \rangle$. If $i = -j$, then this amounts to showing that $\langle a^i \rangle = \langle a^{-i} \rangle$. But this is obvious since an element and its inverse always generate the same subgroup of a group.

    Conversely, suppose that $\langle a^i \rangle = \langle a^j \rangle$. Then it follows that $a_i \in \langle a^j \rangle$ and $a^j \in \langle a^i \rangle$. The fact that $a_i \in \langle a^j \rangle$ implies that we may write $a^i = a^{jm}$ for some $m \in \mathbb{Z}$. Similarly, the fact that $a^j \in \langle a^i \rangle$ implies that we may write $a^j = a^{in}$ for some $n \in \mathbb{Z}$. By Theorem 4.1, we may conclude that $i = jm$ and $j = in$. These equations combine to yield $i = jm = inm$. This forces $nm = 1$ (I'll let you worry about the possibility that $i = 0$), and since $m, n \in \mathbb{Z}$, we have either that $m = n = 1$ or $m = n = -1$. The case $m = n = 1$ forces $i = j$, and the case $m = n = -1$ forces $i = -j$.

30. **Problem:** Suppose a and b belong to a group G, a has odd order, and $aba^{-1} = b^{-1}$. Show that $b^2 = e$.

    Solution. If we take the inverse of both sides of $aba^{-1} = b^{-1}$, we obtain $b = ab^{-1}a^{-1}$. Thus by substituting either $b^{-1} = aba^{-1}$ or $b = ab^{-1}a^{-1}$, we obtain

$$\begin{aligned} b &= a(b^{-1})a^{-1} = a(aba^{-1})a^{-1} = a^2(b)a^{-2} \\ &= a^2(ab^{-1}a^{-1})a^{-2} = a^3(b^{-1})a^{-3} \\ &= a^3(aba^{-1})a^{-3} = a^4ba^{-4}. \end{aligned}$$

I am too lazy to do it here, but one can easily prove (by induction) that for every odd integer m, we have $b = a^m b^{-1} a^{-m}$. In particular, this holds if m is the order of a (which is assumed to be odd). But in that case, this becomes $b = eb^{-1}e = b^{-1}$. From $b = b^{-1}$, it is easy to see that $b^2 = e$.

54. **Problem:** Let $a, b \in G$, and suppose $|a|$ and $|b|$ are relatively prime. Show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

    Solution. Let $x \in \langle a \rangle \cap \langle b \rangle$. Since $x \in \langle a \rangle$, general facts about cyclic groups imply that $|x|$ is a divisor of $|a|$. Similarly, since $x \in \langle b \rangle$, we have that $|x|$ is a divisor of $|b|$. But now $|x|$ is a common divisor of the two relatively prime integers $|a|$ and $|b|$, so it must be that $|x| = 1$. The only element with order 1 is the identity element, so $x = e$.