Homework 0. **Due Thurs Sept 10**

1. If A is an $n \times n$ matrix (with real number entries), a matrix M is called an *inverse* of A of $AM = I_n = MA$, where I_n is the $n \times n$ identity matrix. Prove that an $n \times n$ matrix A either does not have an inverse or has exactly one inverse.

2. Suppose f and g are functions such that the composition $f \circ g$ is defined. Prove that if f and g are both onto, then $f \circ g$ is onto.

3. Let $\mathbb{Q}$ denote the set of rational numbers (so that $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0\}$. Let $\mathbb{Q}_{\geq 0}$ denote the set of nonnegative rational numbers. Prove that the function $f : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ given by $f(x) = x^2$ is not onto. Note: There is a well known fact about square roots that you are welcome to cite (without proving it) if it is helpful.

4. Let $\mathcal{A} = \{8k + 3 \mid k \in \mathbb{Z}\}$ and let $\mathcal{B} = \{8k - 13 \mid k \in \mathbb{Z}\}$. Prove that $\mathcal{A} = \mathcal{B}$.

5. Let $\mathcal{C} = \{m^2 \mid m \text{ is an odd integer}\}$ and $\mathcal{D} = \{4k + 1 \mid k \in \mathbb{Z}\}$. Prove that $\mathcal{C} \subseteq \mathcal{D}$.

Some suggestions on how to begin

1. Suppose that M and N are inverses of A. Then by definition, $MA = I_n = AM$ and $NA = I_n = AN$.

$\cdots$ blah, blah, blah $\cdots$

Therefore $M = N$.

2. Write $f : B \to C$ and $g : A \to B$, so that $f \circ g : A \to C$. To show that $f \circ g$ is onto, let $c \in C$. Then

$\cdots$ blah, blah, blah $\cdots$

Therefore there exists $a \in A$ such that $(f \circ g)(a) = c$, and it follows that $f \circ g : A \to C$ is onto.

3. To show that $f : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ is not onto, we give a specific element of $\mathbb{Q}_{\geq 0}$ that is not the image of an element of $\mathbb{Q}$. If there exists $x \in \mathbb{Q}$ with $f(x) = \cdots$, then blah, blah, blah $\cdots$

This cannot be the case, so we conclude that $f : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ is not onto.

4. We first show that $\mathcal{A} \subseteq \mathcal{B}$. Let $a \in \mathcal{A}$. Then by definition $a = \ldots$.

Blah, blah, blah ....

Therefore $a \in \mathcal{B}$.

We next show that $\mathcal{B} \subseteq \mathcal{A}$. If $b \in \mathcal{B}$, then by definition, $b = \ldots$. Then

blah, blah, blah $\cdots$.

Therefore $b \in \mathcal{A}$.

5. To show that $\mathcal{C} \subseteq \mathcal{D}$, let $c \in \mathcal{C}$. Then by definition, $c = m^2$, where $m \ldots$

Then blah, blah, blah ...

Therefore $c \in \mathcal{D}$.