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# A COGNITIVE GAP BETWEEN ARITHMETIC AND ALGEBRA[1,2]

ABSTRACT. Serious attempts are being made to improve the students' preparation for algebra. However, without a clear-cut demarcation between arithmetic and algebra, most of these undertakings merely provide either an earlier introduction of the topic or simply spread it out over a longer period of instruction. The present study investigates the upper limits of the students' informal processes in the solution of first degree equations in one unknown prior to any instruction. The results indicate the existence of a *cognitive gap* between arithmetic and algebra, a cognitive gap that can be characterized as *the students' inability to operate spontaneously with or on the unknown*. Furthermore, the study reveals other difficulties of a pre-algebraic nature such as a tendency to detach a numeral from the preceding minus sign in the grouping of numerical terms and problems in the acceptance of the equal symbol to denote a decomposition into a difference as in $23 = 37 - n$ which leads some students to read such equations from right to left.

## INTRODUCTION

The question regarding the demarcation between arithmetic and algebra is of more than just academic interest. Teachers who want to prepare students for an eventual course in algebra need to have some idea of what such a demarcation might be for otherwise they are bound to present them with an earlier introduction to algebra rather than with a preparation to algebra that might enhance their readiness. The pupils' lack of readiness may explain the dismal results achieved in algebra in our secondary schools. The best known data which reflect these poor results originate from the United States where a National Assessment of Educational Progress is carried out systematically every four years. In their Second Mathematics Assessment, Carpenter et al. (1981) reported that 67% of the 17-year-olds in their sample had completed a first course in algebra and that only 35% had taken at least a half year of a second course. Eight years later, Swafford and Brown (1989) reported that for the same age group, 75% had completed first-year algebra while 40% had completed a second year. Thus, only about half the students emerge from a first course in algebra, traditionally involving one unknown, with sufficient motivation to enroll in a course dealing with algebra in two variables. We should also keep in mind that these results do not take into account the number of high school drop outs. Considering that even those students who envisage a technical vocation need to have some knowledge of algebra, one can only conclude that many will not be prepared by the end of secondary school.

Possible explanations for the difficulties that students experience with algebra involve both the pace at which it is covered and also the formal approach often used in its presentation. More importantly, it seems that many teachers and textbook authors are unaware of the serious cognitive difficulties involved in the learning

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of algebra. As a result, many students do not have the time to construct a good intuitive basis for the ideas of algebra or to connect these with the pre-algebraic ideas they have developed in primary school; they fail to construct meaning for the new symbolism and are reduced to performing meaningless operations on symbols they do not understand.

Failure in algebra may not reflect the students' learning potential but may simply be induced by a certain type of instruction. Thus, questions regarding the cognitive accessibility do arise: Is algebra within the reach of most students? In answer to this question, the National Council of Teachers of Mathematics recently published Algebra for Everyone (Edwards, 1990) and, as the title implies, it argues that all students ought to be introduced to this subject. This is not a vacuous discussion since in reality a large group of students is prevented from entering algebra. As shown by Lodholz (1990) there are three streams of varying mathematical content by the end of Grade 5 in primary school:

> First we find those students who master the traditional curriculum ... and proceed to a course called pre-algebra in grade 7. Next we find students not quite as successful, and they spend the middle grades reviewing some arithmetic in more complex exercises while they wait to enroll in algebra in the ninth grade. Finally, we observe the unsuccessful students, who stay in school and are relegated to a complete review of arithmetic. Typically, these students will never enroll in a course called algebra.
> ... The data in the recent international assessment (Travers, 1985) indicates that about 10 percent of the students in the United States enroll in algebra by grade 8 and that about 65 percent are in a regular track in algebra by grade 9. The concern for guaranteeing success in algebra for all students is then directed towards the lower one-fourth of the student population, who presently do not even think about algebra.
> (Lodholz, 1990, 24–25)

We are thus concerned with two main groups of students: those who enter Algebra 1 and fail or encounter major difficulties, and those who do not even enroll in an initial course. The first group does reasonably well in arithmetic but experiences problems with algebra. For them, the traditional curriculum in arithmetic may not be sufficient and a bridge between arithmetic and algebra has to be constructed. The second group is rejected on the basis of poor performance in arithmetic. The assumption here is that elementary arithmetic is a necessary and sufficient prerequisite. Nevertheless, questions arise as to whether or not they should be exposed to some other mathematical experience, such as pre-algebra, instead of just repeating their unsuccessful one in arithmetic.

In order to achieve better results and reach a greater population, several educators have raised the possibility of spending some of the time allocated to mathematics in the seventh grade to introduce pre-algebra, that is, the transition phase between arithmetic and algebra. One such example was the ALGEBRIDGE project initiated by Educational Testing Service and the College Entrance Examination Board (1990) whose objective was to bridge the gap between arithmetic and algebra. Several other books on pre-algebra have been published to date. The idea of introducing pre-algebra in seventh grade has been well received by mathematics teachers since in this first year of secondary school most of the time is spent on reviewing primary school mathematics which they may feel does not provide, in its present form, a solid foundation for algebra. However, too often the pre-algebra textbooks used merely consist of an earlier introduction to algebra and simply spread out over a longer period of time the topics presented in the usual formal course. While these attempts are successful with some students, the lack of awareness of the cognitive obstacles that need to be overcome in the transition between arithmetic and algebra clearly limit their pedagogical scope.

Trying to find a demarcation line between arithmetic and algebra is not a trivial problem. Some people tend to see algebra everywhere and claim that children solving a missing addend problem such as $4 + \square = 9$ are doing algebra since the place holder can be considered as an unknown. Of course, this can be solved using purely arithmetic means such as counting procedures or an inverse operation. At the other extreme we find educators who claim that algebra cannot be introduced without a prior knowledge of integers. This would have come as a surprise to al-Khwarizmi, considered to be the father of algebra (Boyer/Merzbach, 1991), who introduced it to the Mediterranean world around the ninth century at a time when negative integers were not yet considered to be numbers.

A most interesting idea regarding this demarcation has been presented by Filloy and Rojano (1984) who suggested that one can find a sharp delineation between arithmetic and algebra, a "*didactic cut*", when the unknown occurs on both sides of a first degree equation in one unknown as in $ax + b = cx + d$. The students who were studied in their research were subjected to some instruction regarding the solution of equations involving a single occurrence of the unknown: they were taught to use inverse operations. Afterwards, in order to solve equations with the unknown on each side of the equal sign, they were introduced to the classical balanced scale model and another original model based on the equivalence of distinct rectangular areas (Filloy, 1987; Filloy and Rojano, 1989, 1985a, b; Gallardo and Rojano, 1987).

It is difficult to judge from this research how seventh graders might respond to the task of solving equations *prior to any* instruction in algebra. Yet, in trying to determine some differentiation between arithmetic and algebra, knowledge of the range of equations that can be solved by students using purely intuitive means might prove to be revealing. Of course, knowledge of this range can only supply part of the answer. More importantly, the solution procedures used must be assessed. In fact, this is a main drawback in Filloy and Rojano's notion of a didactic cut. They define it in terms of a mathematical form, that of a class of equations with the unknown on each side of the equal sign, without any reference to the mathematical processes involved in their solution. This raises some difficult questions. For instance, would a student solving such equations by systematically substituting numerical values be doing algebra or arithmetic? If the unknown occurs twice but on the same side of the equal sign, would the student's solution procedure differ?

In fact, we cannot look at an equation without also determining how the letter representing the unknown is perceived by the learner. In a paper on the development of formal reasoning, Collis (1975) investigated this problem in the context of algebraic expressions. He called these letters *pronumerals* since they stand for some number or set of numbers and described three distinct levels of conceptualization:

> *Lowest level* (10–11 year-olds): maps pronumeral directly into a specific number which seems viable at the time – if one trial does not give a satisfactory result, child gives up working on that item.
> *Middle level* (12–13 year-olds): those willing and able to map a group of numbers (each specific on any occasion) onto the pronumeral in a sort of "guessing and testing" technique.
> *Top level* (14–15 year-olds): seem to have extracted a concept of "generalised number" by which a symbol "$b$" (say) could be regarded as an entity in its own right but having the same properties as any number with which they had previous experience.
> (Collis, The Development of Formal Reasoning, pp. 43–44)

While the ideas presented by Collis are very interesting, his age levels must be taken with some caution for the algebraic expressions used in his assessments were formal and somewhat artificial. Nevertheless, the idea of a pronumeral evolving into a generalized number is quite enlightening. However, it is not sufficient to endow it with "the same properties as any number", for this can be interpreted quite passively, as for example "let $n$ be an even number". In fact, the pronumeral must also be endowed with the operational properties of number; the unknown must be perceived as a generalized number that can be subjected to all the operations performed with and on numbers. Perhaps the expression "*operational generalized number*" describes this necessary evolution.

That this additional property is far from trivial has been evidenced in some prior research. For instance, clinical evidence of this problem appears in Davis' interview of an exceptionally bright 12-year-old student (Henry), enrolled in an experimental class following an enriched algebra program (Davis, 1975). After being taught how to solve equations such as $3x + 2 - 5x + 6 = 9x - 3x + 23$, the class was introduced to rational forms such as $3/x = 6/(3x + 1)$. Upon being told to multiply both sides by $x$, Henry seemed to agree that by multiplying $3/x$ by $x$ he would be left with 3. However, when told to multiply the right-hand side by $x$, he replied: "*How can we multiply by x when we don't know what x is?*" Although this kind of evidence is anecdotal, it is worth mentioning for it so clearly indicates the cognitive problem at hand. Although Henry may view the literal symbol as a generalized number, he cannot operate with it.

More substantial evidence can be gathered from a large scale assessment study carried out in Great Britain involving 3000 secondary school students in their second, third or fourth year aged 13.3, 14.3, and 15.3 respectively (Kuchemann, 1981, 1978). Only results obtained from the 14-year-olds in their second year of algebra have been published. When asked to *add 4 onto 3n*, 36% gave a correct answer $3n + 4$, but 31% answered $7n$ while 16% gave 7 as the answer. When asked to *multiply* $n + 5$ *by 4*, 17% gave correct answers $4n + 20$ or $4(n + 5)$, while 19% answered $4n+5$ or $4 \times n + 5$, 31% gave $n + 20$ as the answer, and 15% simply wrote 20. The last two results indicate that at least 46% of the students will not perform the required operation on the literal symbol.

Of course, some of the difficulties faced by the British students are specific to algebraic expressions. One cognitive problem identified with this mathematical form is what Davis (1975, p. 18) called the "name-process" dilemma by which an expression such as $6x$ is both an indication of a process ("What you get when you multiply 6 by $x$") and a "name for the answer". Sfard and Linchevski (1993) have suggested that the term "process-product dilemma" better describes this cognitive problem. A somewhat different slant is provided by Collis' theory of the student's Acceptance of the Lack of Closure ($ALC$) which describes the level of closure at which the pupil is able to work with operations (Collis, 1974). He observed that at the age of seven, children require that two elements connected by an operation (e.g. $3 + 2$) be actually replaced by a third element; from the age of 10 onwards, they do not find it necessary to make the actual replacement and can also use two operations (e.g. $6+4+5$); twelve year-olds can refrain from actual closure and are capable of working with formulas such as $Volume = L \times B \times H$; between the ages of 13 and 15, although students are not yet able to handle variables, they have no difficulty with symbolization as long as the concept symbolized is underpinned by a particular concrete generalization. Collis' $ALC$ theory is particularly relevant to the teaching of algebraic expressions since the operations performed on the pronumerals cannot be closed as in arithmetic. This problem was evidenced in a teaching experiment aimed at constructing meaning for algebraic expressions (Chalouh and Herscovics, 1984, Herscovics and Chalouh, 1984). Even after instruction, some students could not accept $8 \times a$ as the area of an indicated rectangle unless it was inserted in the formula "Area of rectangle$= 8 \times a$".

Thus, if even after an introduction to algebra, students experience difficulties in performing operations with or on a letter representing an unknown or a generalized number, one can hardly expect them to do so spontaneously without any instruction. Although the letter in an equation or an algebraic expression may have a numerical referent in the pupil's mind, this does not necessarily render it operational. Meaning for these operations with literal symbols still has to be constructed. In fact, it is to supply such meaning that Filloy and Rojano felt the need to provide some specific instruction for the solution of equations with the unknown occurring on both sides of the equal sign.

The inability to operate spontaneously with or on the unknown indicates the existence of a *cognitive gap* that can be considered a demarcation between arithmetic and algebra. Given the students' difficulty with the Acceptance of the Lack of Closure of algebraic expressions, algebraic equations seem to be more appropriate for any further investigation of this cognitive gap. It can be conjectured that such a cognitive gap would establish an upper boundary to the scope of the student's informal equation solving procedures. These informal procedures are limited to inverse operations or numerical substitutions and approximations for equations with a single occurrence of the unknown. For a double occurrence of the unknown, whether on the same side or on each side of the equal sign, little is known about the students' spontaneous handling of these problems but our hypothesis would limit the student to numerical substitution processes.

It is to verify this hypothesis that we designed an experiment consisting of two parts. The first part dealt with the assessment of arithmetic concepts and skills that are considered to be essential for the solution of equations such as the cancellation of inverse operations and the order of operations, or arithmetic concepts that need to be expanded such as the meaning of the equal sign. The second part was a systematic assessment of the range of equations that seventh graders could solve without prior instruction as well as a survey of the spontaneous solution processes they used.

## METHODOLOGY

We decided to interview a whole class of seventh graders in order to observe a wider range of mathematical abilities. Of the 27 students in the class, 22 were given parental authorization and were interviewed individually in two 45-minute session. Nine of these students (mostly the weaker ones) needed a third session lasting from 15 to 30 minutes. The average age of the 22 students was 12 years and 9 months. The class was chosen in a parochial school in Montreal on the basis of easy access, a very cooperative staff that allowed individual students to be interviewed whenever needed, and especially, a classroom teacher who collaborated with us in avoiding any introduction of algebra prior to our investigation (except for the meaning of concatenation and numerical substitutions for literal symbols in the context of formulas) and also in making all the necessary arrangements. The classroom teacher provided us with the results of the students' performance in school mathematics in their regular tests and exams: 3 students had results better than 90%, 8 students were in the 80% range, 5 in the 70% range, 5 in the 60% range, and 1 in the 50% range. Thus the class could be divided into two groups, 11 achieving 80% or better and 11 between 50 and 80%.

In order to focus the students' attention on the equation solving processes, we wished to eliminate most of the arithmetic drudgery and wanted also to avoid the pupils' problem of keeping track of their various attempts. Hence, students were strongly encouraged to use a calculator provided during the interview, a very simple calculator that did not take into account the order of operations. Each equation was presented on half a page with ample space for the students to write whatever they wanted. Students were asked to "think aloud" and the interviewer, sitting right next to the subject, wrote down each attempted solution on the side of the student's worksheet. The student was free to consult the interviewer's notes at all times. Students were required to go through a *verification* procedure: once they thought they had found a solution, they had to write it above the unknown in the equation and verify its validity on the calculator. Thus, an incorrect solution would induce the search for a correct one. An observer was present during each interview in order to take notes and also to participate later in the analysis of the student's responses. In fact, the observer participated actively by helping the interviewer keep track of the semi-standardized questionnaire and by intervening whenever he or she felt that the interviewer was misunderstanding the subject.

*Part 1: Preliminary Assessment*

The function of the preliminary assessment was to gather some relevant information about each student regarding his or her knowledge of the arithmetic concepts and skills needed for the solution of equations. The first question was *"Do you know the word equation"? "Can you show me an example?"*.

Nearly all our subjects responded with examples involving a numerical operation written vertically. When asked to write horizontally, most of them wrote operations such as $5 + 7 =$ . In order to allow the students to become familiar with the provided calculator, they were next asked to evaluate four arithmetic operations involving large numbers.

The second set of questions verified their acceptance of the *different uses of the equality symbol*. This problem had been investigated by Behr, Erlwanger, and Nichols (1976) who had shown that most primary school pupils viewed the equal sign operationally in the sense of indicating the need to perform the required operation and writing the result as in $3 + 5 =?$ However, when faced with a *decomposition* such $9 = 4 + 5$, many children would refuse to accept it, claim that it was written backwards and re-write it as $4 + 5 = 9$. They experienced further difficulties when the equal sign was used to represent an arithmetic *equivalence* such as $3 + 7 = 6 + 4$. Many would simply break it up and write two equations: $3+7 = 10$ and $6+4 = 10$. Several researchers have pointed out that such a limited meaning of the equal sign persisted even among some students in secondary school (Cortes, Vergnaud and Kavafian, 1990; Herscovics and Kieran, 1980; Kieran, 1981a).

We asked our students if $34 = 19 + 15$ was written correctly or incorrectly. Two of our weaker students did not accept the use of the equal symbol for a decomposition into a sum. In addition, during the latter part of the assessment dealing with the solution of equations, we found that when the equations represented a decomposition, 7 students ended up reading them from right to left. This was of no consequence when only a sum was involved as in $35 = n + 16$. However, in the case of a subtraction such as $364 = 796 - n$, three students changed the intended problem by reading it from right to left "$n$ minus 796 equals 364".

To verify our students' acceptance of the equal sign as a symbol of arithmetic equivalence, we asked them if $15 + 7$ was equal to $10 + 12$ and then how they would show that they were equal. Most of them wrote down two equations indicating that each sum added up to 22. Yet, when asked if we could express it as $15 + 7 = 10 + 12$, all of them agreed.

The third set of questions assessed the students' familiarity with the *order of operations*. Prior research (Kieran, 1979) had shown that when solving a string of arithmetic operations, students tended to perform them sequentially, one at a time, going from left to right. This tendency might become an obstacle when the order of operations had to be observed. We investigated this by asking our subjects to evaluate $5 + 6 \times 10 =?$ $17 - 3 \times 5 =?$ and $8 \times (5 + 7) =?$ Out of 22 students, 17 (77%) gave 110 as the answer to the first string of operations, indicating that they had first performed the addition. Regarding the second string, only 3 of them performed the subtraction first, the other 19 immediately finding the correct answer 2. No errors were evidenced in the use of brackets. Students were then challenged by the interviewer who referred back to the first string by explaining that "another student had performed the multiplication first, found 60, and then added it to 5 to obtain 65. Who was right, you, the other student, or are you both right?" At that point all students remembered that they had been taught the order of operations and had learned the mnemonic BOMDAS (Brackets first, or Multiplication and Division, then Addition or Subtraction). They immediately corrected their initial mistake and went on to evaluate a fourth string, $6 + 5 \times 4 + 7 \times 3 =?$, correctly.

Another element we wanted to assess was the student's ability to perceive a string of operations more globally, by somehow distancing themselves from the string and taking so to speak an *overview* of it. We felt that this could be related to a more deliberate approach to equation solving. One indicator of a global view is the students' perception of cancellation. We asked them "*Can you work out* $17 + 59 - 59 + 18 - 18 =?$" Only 5 students (three of them among the weak ones) saw that no operation had to be performed because of the cancellations, 4 more students perceived only one of the two cancellations, but 13 (59%) did not realize at all that they were present and proceeded to perform all the operations sequentially. The interviewer pointed out the cancellations and presented another string: $237 + 89 - 89 + 67 - 92 + 92 =?$ Results were somewhat better with 8 students perceiving both cancellations, 8 more perceiving the first one ($+89 - 89$), but not the second one ($-92 + 92$), and still 5 subjects not being aware of any cancellation.

The final part of the preliminary assessment ascertained the students' familiarity with *concatenation*, that is, the juxtaposition of the letter and the numeral as in "$3n$". In their mathematics textbook the topic of symbols and substitution had been introduced early on. The students were told

> "... $15 + n$ is called a variable expression. Symbols, such as the letter $n$, are called variables or placeholders. Often □ or other letters such as $x$, $y$, $b$, etc. are used as placeholders. (Remember, each variable represents some number). ... Variables are combined with the operations $+$, $-$, $\div$, and $\times$ to construct variable expressions. $m + 5m$, $5y - y$, $p \div 2$, $2 \times a$ (usually written $2a$). (Ebos, Robinson, Tuck, 1984, Math is/2, Sec. edition, p. 27)

Hence not surprisingly, all our students knew that $3n$ represented $3 \times n$ and when asked what they would get if they replaced $n$ by 2 or 5, all of them answered 6 and 15, respectively. This is in sharp contrast with the results obtained with another group of students who had previously been exposed to concatenation in an introduction to algebra, but in a post test reverted back to an arithmetical frame of reference and therefore thought that the substitutions would yield 32 and 35 respectively (Herscovics and Chalouh, 1985).

Finally, we questioned our students about their preferred form of an equation. Carpenter et al. (1981) had found that among 13-year-olds, 91% succeeded in solving an equation such as $4 \times \square = 24$ but that only 65% succeeded when the numeral and letter were concatenated as in $6m = 36$. Among our 22 students, 12 chose to work with the concatenated form ($3n$), 8 chose the form using the multiplication sign ($3 \times n$), and 2 preferred to work with placeholders ($3 \times \square$).

### *Part 2: Range of Equations and Solution Processes*

As mentioned earlier, Filloy and Rojano's research on the students' solution of equations involved prior instruction. Thus one cannot infer from it how students might handle equations without any formal instruction. Although partial answers to this question have been provided by Kieran (1984, 1981b), both her number of subjects (6 and 10 respectively) and the number of equations tested (14 and 9 respectively) were too small to bring out any general trends. In prior assessments dealing with this topic, little attention was paid to equations in which the unknown appeared as the subtrahend or the divisor (e.g. $273 - n = 164$; $525 \div n = 15$) or to equations in which *the unknown appeared twice but on the same side* of the equality symbol (e.g. $11n + 14n = 175$). Although Kieran used one of each type in her work, her results did not indicate whether or not the students operated *on or with* the unknown.

### *Selection of Equations*

Knowing the difficulties that students experience with rational numbers, we decided to restrict our equations to natural numbers. Thus, the obstacles encountered by our subjects would be of an algebraic nature, not due to a weakness in arithmetic. We wanted to submit to our students equations involving all *four operations*, as well as variations in the *position of the unknown*. It was conceivable that when faced with equations in which the unknown was the subtrahend or the divisor, the students might solve them by operating with or on the unknown. The *size of numbers* was also taken into account in order to determine if we would witness a shift in solution strategies. This was a reasonable assumption since in many cases, students will use primitive processes on equations they find very easy and use their more sophisticated procedures when they feel they are warranted. Whenever the same type of equation was repeated, the one with the smaller numbers was presented first in order not to discourage the subject.

Another variable that was taken into account was the *direction of the equation*. Prior to our preliminary assessment, we had no idea how well the students accepted the equality symbol for the representation of either an arithmetical decomposition into an operation or an arithmetical equivalence. We also varied the *number of operations* to verify one of Kieran's early results (Kieran, 1984) showing that some students were having difficulties in solving an equation involving several arithmetic operations (e.g. $4 + n - 2 + 5 = 11 + 3 - 5$). We included in our list

TABLE I

Coding of solution procedures

| | |
|---|---|
| $C$ | = Complementary subtrahend or factor |
| $Div.2$ | = Divides by 2 |
| $G$ | = Grouping of numerical terms |
| $GU$ | = Groups terms in the Unknown |
| $I$ | = Inverse operation |
| $NF$ | = Number Fact |
| $RS$ | = Random Substitution |
| $SS$ | = Systematic Substitution |
| $IAA$ | = Inverse Addition Algorithm |
| $ISA$ | = Inverse Subtraction Algorithm |
| $SFT$ | = Substitutes and succeeds on First Trial |
| $S+Ap$ | = Substitution and approximation |

8 equations in which the *order of operations* used by the students was specifically examined.

The last and most critical area of our investigation was the testing of equations in which the same *unknown occurred twice*. We included 10 equations in which the unknown appeared *on the same side* of the equality symbol in order to assess if students would group the terms involving the unknown and thereby exhibit some ability to operate with or on the unknown. We also included 3 equations in which the unknown appeared on *both sides* of the equal sign since 5 or sometimes all 6 of Kieran's pupils had not been able to solve such equations (Kieran, 1984).

## RESULTS

Since all equations were solved by *nearly all* our students, we will focus our attention on the solution procedures that proved to be the most frequently used as well as on the specific difficulties encountered by the students with some of the equations. Our very high success rate might be due to optimal conditions mentioned earlier. Table I describes the coding used to identify the different solution procedures observed during the interviews.

### *Coding*

The code $C$ represents the procedure used in solving an equation like $37 - n = 18$, where most students performed $37 - 18 = 19$. This procedure uses the difference as a subtrahend in the complementary form $37 - 19 = 18$. We describe it as using the Complementary subtrahend. A similar interpretation was applied to equations in which the unknown was the divisor.

TABLE II
Equations involving addition and subtraction

| Equation | Success rate | Most frequent procedures | | | | | |
|---|---|---|---|---|---|---|---|
| (1) $14 + n = 43$ | 100% | $I$ | 82% | $S + Ap$ | 9% | $IAA$ | 4% |
| (2) $35 = n + 16$ | 100% | $I$ | 77% | $S + Ap$ | 14% | $IAA$ | 4% |
| (3) $n + 596 = 1282$ | 100% | $I$ | 86% | $S + Ap$ | 9% | $IAA$ | 4% |
| (4) $437 + n = 984$ | 100% | $I$ | 87% | $S + Ap$ | 9% | $IAA$ | 4% |
| (5) $1269 = 693 + n$ | 100% | $I$ | 87% | $S + Ap$ | 9% | $IAA$ | 4% |
| (6) $37 - n = 18$ | 100% | $C$ | 73% | $SS$ | 14% | $NF$ | 4% |
| (7) $n - 13 = 24$ | 100% | $I$ | 77% | $S + Ap$ | 9% | $ISA$ | 9% |
| (8) $17 = n - 15$ | 100% | $I$ | 77% | $S + Ap$ | 9% | $ISA$ | 9% |
| (9) $23 = 37 - n$ | 95% | $C$ | 72% | $S + Ap$ | 14% | $ISA$ | 9% |
| (10) $273 - n = 164$ | 95% | $C$ | 72% | $S + Ap$ | 14% | $ISA$ | 9% |
| (11) $n - 872 = 167$ | 95% | $I$ | 77% | $S + Ap$ | 9% | $ISA$ | 9% |
| (12) $235 = n - 163$ | 100% | $I$ | 86% | $SS$ | 4% | $S + Ap$ | 4% |
| (13) $364 = 796 - n$ | 91% | $C$ | 73% | $S + Ap$ | 9% | $ISA$ | 9% |

TABLE III
Equations involving multiplication and division

| Equation | Success rate | Most frequent procedures | | | | | |
|---|---|---|---|---|---|---|---|
| (14) $16n = 64$ | 100% | $I$ | 68% | $NF$ | 18% | $SS$ | 9% |
| (15) $32n = 928$ | 100% | $I$ | 82% | $SS$ | 18% | | |
| (16) $2088 = 174n$ | 100% | $I$ | 82% | $SS$ | 18% | | |
| (17) $84 \div n = 4$ | 100% | $C$ | 78% | $SS$ | 18% | $I + NF$ | 4% |
| (18) $525 \div n = 15$ | 100% | $C$ | 77% | $SS$ | 23% | | |
| (19) $n \div 6 = 13$ | 100% | $I$ | 85% | $SS$ | 11% | $NF$ | 4% |
| (20) $n \div 8 = 57$ | 100% | $I$ | 86% | $SS$ | 14% | | |
| (21) $15 = n \div 7$ | 100% | $I$ | 86% | $SS$ | 14% | | |
| (22) $23 = 115 \div n$ | 95% | $C$ | 72% | $SFT$ | 14% | $SS$ | 9% |

Inverse forms of the addition algorithm ($IAA$) and the subtraction algorithm ($ISA$) were used by a few students. For instance, in solving $1269 = 693 + n$, they would write it vertically and work out the missing addend digit by digit.

$$\begin{array}{r} 693 \\ +\,\text{---} \\ \hline 1269 \end{array}$$

TABLE IV

Equations involving the grouping of numerical terms

| Equation | Success rate | Most frequent procedures | | | | | |
|---|---|---|---|---|---|---|---|
| (23) $n + 15 + 27 = 61$ | 100% | $G + I$ | 63% | $I + I$ | 18% | $G + NF$ | 9% |
| (24) $n + 34 = 29 + 38$ | 100% | $G + I$ | 73% | $G + NF$ | 14% | $G + Ap$ | 4% |
| (25) $14 + n + 17 = 50$ | 100% | $G + I$ | 68% | $I + I$ | 23% | $SS$ | 4% |
| (26) $23 + n + 18 = 44 + 16$ | 100% | $G + G + I$ | 64% | $G + I + I$ | 18% | $G + NF$ | 14% |
| (27) $n + 15 - 9 = 61$ | 100% | $G + I$ | 77% | $G + I + I$ | 4% | $G + IAA$ | 4% |
| (28) $39 + n - 12 = 74$ | 95% | $S + Ap$ | 41% | $I + I$ | 32% | $G + I$ | 18% |
| (29) $4 + n - 2 + 5 = 11 + 3 - 5$ | 50% | $G + SS$ | 37% | $G + I$ | 9% | $G + NF$ | 4% |

### *Equations with Only One Operation*

Tables II and III show the procedures used to solve equations involving one operation. The results indicate that essentially two types of procedures are used.

For equations in which only addition or multiplication are present, students will overwhelmingly revert to solving these by using the inverse operation, but the numbers in the equation must be large enough to go beyond known number facts. These results are very similar to those obtained by Kieran (1981) but differ from those of Gallardo and Rojano (1987) who found "some confusion between the various operations, interpreting addition as subtraction, subtraction as division" which might explain some of their students' problems with inverse operations.

For equations involving only subtraction or only division, one must distinguish between cases where the unknown is a minuend or a dividend from those cases where it is a subtrahend or a divisor. In the former cases, over 77% of the students use the respective inverse operations. In the latter cases, students respectively subtract the given difference from the given minuend or divide the given dividend by the given quotient. At no time did we see any evidence of students directly performing operations on or with the unknown. Thus we can conclude that students solve these equations by *working around the unknown* at a purely numerical level.

Some difficulties are worth mentioning. Five students read some of the following equations from right to left (eq. 9, 10, 11, 13 and 22). Two students had some problem in accepting the equal sign as a symbol for decomposition and felt the need to re-write equation 9 as $37 - n = 23$, and did so again with equations 16, 21 and 22.

### *Grouping of Numbers*

The presence of several numerical additions or subtractions in an equation does not seem to have any major impact except in one special case. Table IV shows that in general, students have no problems spontaneously simplifying the given

TABLE V
Equations involving both additive and multiplicative operations

| Equation | Success rate | Most frequent procedures | | | | | |
|---|---|---|---|---|---|---|---|
| (30) $4n + 17 = 65$ | 100% | $I + I$ | 41% | $I + NF$ | 36% | $SS$ | 14% |
| (31) $13n + 196 = 391$ | 100% | $I + I$ | 77% | $SS$ | 14% | $I + SS$ | 4% |
| (32) $3n + 12 = 33$ | 100% | $I + I$ | 50% | $SS$ | 27% | $I + NF$ | 14% |
| (33) $16n - 215 = 265$ | 100% | $I + I$ | 73% | $SS$ | 18% | $I + SS$ | 4% |
| (34) $420 = 13n + 147$ | 100% | $I + I$ | 73% | $SS$ | 18% | $I + SS$ | 4% |
| (35) $188 = 15n - 67$ | 95% | $I + I$ | 68% | $SS$ | 23% | $NF$ | 4% |
| (36) $6 + 9n = 60$ | 100% | $I + I$ | 41% | $I + NF$ | 18% | $NF$ | 14% |
| (37) $63 - 5n = 28$ | 86% | $C + NF$ | 41% | $C + I$ | 27% | $SS$ | 18% |

equations, the majority performing the indicated numerical operations. However, a comparison between equations 25 and 28 involving an unknown between two numerical terms indicates that the presence of a subtraction induces a major shift in the choice of solution procedures.

It is equation 29 which gave us the most startling results. Nevertheless, they compare with those of Kieran (1984) who found that 3 of her 6 subjects provided incorrect solutions. In our study, 11 of our 22 students failed (5 in the upper half and 6 in the lower half of the class) and did so as a result of grouping the numbers on the left incorrectly. They ignored the subtraction sign preceding 2 and simply focused on the posterior addition sign, *adding* $2 + 5$! Having thus obtained 7 they ended up solving $4 + n - 7 = 9$ and found $n = 12$ as their solution. Even after inserting the solution in the equation in order to verify, they maintained their erroneous grouping and hence were not in a position to detect their mistake. This unexpected detachment of a number from the subtraction symbol preceding it also occurred in our preliminary assessment and we will deal with it later on.

### *Combined Additive and Multiplicative Operations*

Regarding equations involving multiplication as well as addition or subtraction, our results differ from prior investigations. For equations with large enough numbers, Kieran (1981) found that only 2 of her 10 subjects were "undoing twice". Filloy and Rojano (1984) found that for two-step equations their high-ability subjects used a mixed strategy, "undoing the operation on the first step and applying 'plugging-in' (non-systematic trial and error strategy) or specific fact procedures on the second one". Our results show that with large enough numbers (equations 31, 33, 34, and 35) the percentage of students using inverse operations consecutively ranged from 68% to 77%. It should be noted that the three students who failed to solve equation 37 were all reading it as $5n - 63 = 28$.

TABLE VI

Double occurrence of unknown on the same side

| Equation | Success rate | Most frequent procedures | | | | | |
|---|---|---|---|---|---|---|---|
| (38) $n + n = 76$ | 100% | $Div.2$ | 68% | $SS$ | 27% | $NF$ | 4% |
| (39) $n + 5 + n = 55$ | 100% | $I + Div.$ | 32% | $NF$ | 27% | $I + NF$ | 14% |
| (40) $3n + 4n = 35$ | 100% | $SS$ | 55% | $SFT$ | 36% | $GU$ | 9% |
| (41) $9n - 4n = 35$ | 100% | $SFT$ | 59% | $SS$ | 36% | $GU$ | 4% |
| (42) $11n + 14n = 175$ | 100% | $SS$ | 82% | $SFT$ | 9% | $GU$ | 4% |
| (43) $17n - 13n = 32$ | 100% | $SS$ | 86% | $GU$ | 9% | $S + Ap$ | 5% |
| (44) $5n + n = 78$ | 100% | $SS$ | 86% | $SFT$ | 14% | | |
| (45) $7n - n = 108$ | 100% | $SS$ | 91% | $SFT$ | 9% | | |
| (46) $7n + 5n + 7 = 55$ | 100% | $SS$ | 55% | $I + SFT$ | 23% | $SFT$ | 9% |
| (47) $3n + 5n + 4n = 19$ | 100% | $SS$ | 36% | $SFT$ | 36% | $I + SFT$ | 18% |

*Double Occurrence of Unknown on the Same Side*

Since none of the students had ever seen equations of this type before, the interviewer had to explain: *When we have the letter (or box) twice in the equation, it means that you have to replace each letter with (or put in each box) the same number.*

Equations 38 and 39 must be considered as trivial cases that were meant to familiarize the students with a double occurrence of the unknown and thus cannot be seen as representative. In comparing Table VI with the previous tables we note a major change in the solution procedures used in solving equations 40 to 47. Whereas some form of substitution ($SS$ or $SFT$) had been used in the solution of previous equations, the most frequent occurrence was in the 20% range (23% for equation 35 and 27% for equation 37). However, in the solution of equations 40 to 47, the frequency of substitution procedures ranges from 87% (equation 46) to 100% (equation 45). These results confirm our conjecture regarding the students' inability to operate with or on the unknown and thus having to rely on numerical solution processes.

It is in this set of equations that we found isolated cases of spontaneous algebraic methods, by which we mean operating with or on the unknown. One student was able to solve equations 40 and 41 by grouping the two terms in the unknown, a second student did so for equations 40 and 43, and a third student solved equations 46 and 47 by first performing an inverse operation and then grouping the unknown. It is noteworthy that none of these students extended the grouping procedure to equations 44 and 45 where one of the terms in the unknown did not indicate a coefficient. The sporadic incidence of this algebraic behavior prevents us from viewing these students as having achieved any stable algebraic knowledge. Instead, the overwhelming evidence points to the fact that grouping of the unknown is not a procedure that is acquired spontaneously.

TABLE VII
Equations with unknown on both sides

| Equation | Success rate | Most frequent procedures | | | |
|---|---|---|---|---|---|
| (48) $n + 15 = 4n$ | 91% | $SFT$ | 59% | $SS$ | 32% |
| (49) $4n + 9 = 7n$ | 86% | $SS$ | 55% | $SFT$ | 31% |
| (50) $5n + 12 = 3n + 24$ | 91% | $SS$ | 86% | $SFT$ | 5% |

*Unknown on Both Sides*

In the last three equations the unknown occurred on each side of the equal sign. Our results differ from those of Kieran (1984) whose 6 subjects came up with incorrect solutions for the equation $4x + 9 = 7x$. This same equation (our equation 49) was submitted to our 22 students and 19 of them found a correct answer either by systematic substitution or by succeeding in their first attempt at substitution. The other two equations in this set were solved by 20 pupils. These equations were more difficult to solve by substitution. Whereas in previous cases the systematic substitution yielded a sequence of approximations to a given number, this was no longer the case when the unknown appeared on both sides. The process of systematic substitution yielded a comparison of the two functions on either side of the equal sign and at some point the difference between the two was inverted and the respective numerical values became increasingly divergent.

*Detachment from the Minus Sign*

As mentioned earlier in the analysis of the procedures used to solve equation 29, we found among some students a tendency to ignore the minus sign preceding the number 2 in $4 + n - 2 + 5 = 11 + 3 - 5$. In fact, *16 of the 22 students* did at one point add 2 and 5. However, 4 of them corrected their initial mistake during the verification of their solution. The detachment of the minus sign was not restricted to instances where a number was subtracted from an unknown, for we found evidence of this in the preliminary assessment on the questions dealing with global perception of a string of operations. For the string $17+59-59+18-18=?$ we found that 6 students were ignoring the minus sign in front of 59 and adding 59 to 18. In the second string $237 + 89 - 89 + 67 - 92 + 92 =?$ we found that 5 students were ignoring the minus sign in front of 89 and instead were adding 89 to 67; but even more students (10) ignored the minus sign in front of 92 and were adding 92 to 92 and then subtracting the sum 184.

The detachment of a number from the preceding minus sign may also explain the very different answers we obtained on the order of operations. It will be recalled that 17 of our 22 students had worked out the string $5 + 6 \times 10 =?$ by first performing the addition but in the case of $17 - 3 \times 5 =?$ only three subjects had performed the subtraction first. Although the two strings have a similar structure, the results obtained are markedly different.

This detachment of the minus sign was somewhat of a surprise to us. The high incidence of this mistake indicates that the problem is not idiosyncratic but may well reflect unsuspected cognitive obstacles (Herscovics, 1989). Perhaps this problem is somewhat induced by the introduction of the order of operations. By learning that multiplication and division take precedence over addition and subtraction, some students may conclude that addition takes precedence over subtraction.

Another plausible conjecture reaches more fundamental issues of a structural nature. Both Booth (1989) and Kieran (1989) have pointed out that students construct their algebraic notions on their previously acquired experience in arithmetic. As such, their algebraic system inherits the structural properties associated with the number systems known to them, Booth (1989) has suggested that "the students' difficulties in algebra are in part due to their lack of understanding of various structural notions in arithmetic". Of course, as seen in the procedures used to solve equation 29, these problems are compounded when the surface structure indicates that a number 2 has to be subtracted from an unknown. Unless the students know that they can use commutativity to obtain $(n + 4) - 2 + 5$ and then use associativity to perform $((4 - 2) + 5)$, they must resort to other means. Since our students had not seen negative numbers, the first term in the expression $-2 + 5$ may not have made any sense to them. The addition of 2 and 5 can then be seen as an attempt to transform the problem into a more meaningful one.

## CONCLUSION

Of course, we do not claim that the cognitive gap we have identified is the only one between arithmetic and algebra. Many others are known to exist. For instance, the difficulties students experience in translating word problems into algebra indicate another major cognitive gap most recently illustrated by MacGregor and Stacey (1993). Another serious gap exists in the translation of functions from a tabular form representation into graphs and equations (Herscovics, 1980, 1979). The study of the latter one becomes even more important in view of the new technological innovations involving graphing calculators and computers.

The objective of this study was to identify a possible demarcation between arithmetic and algebra in the context of equations. To this end, we selected a sequence of 50 first degree equations in one unknown that were to be solved by a class of seventh graders prior to any formal introduction to algebra. It must be remembered that we made every effort to create optimal conditions: a calculator was available to alleviate arithmetical chores; the interviewer's notes were always available to the students thereby eliminating their need to keep track; subjects were always reminded that the correctness of the answer was not our prime concern since we were interested in the procedures they used; each student was interviewed individually and was the focus of attention of two adults, the interviewer and the observer, who were interested in his/her thinking processes. Thus there is no doubt in our mind that our subjects performed better than under normal classroom conditions. Furthermore, the school we had chosen for our study was well known for its emphasis on academic excellence. Since our objective was to discover the upper limit of the pre-algebraic potential of 12-year-olds, this was an appropriate choice.

That our students were in fact ready for an introduction to algebra was evidenced by their repeated inquiries about the existence of more efficient ways to solve the equations with a double occurrence of the unknown. Quite often they had tried to find different procedures by using the given numbers in the equation in what seemed to be random combinations. Since these attempts were seldom successful, they reverted to their "guess and check" method. But their guessing was hardly random. They proved to be very systematic in their approximations.

Our study does bring new light to the upper limits of informal solution processes. The existence of a didactic cut as suggested by Filloy and Rojano (1984, 1987) has not been confirmed. They defined the didactic cut as "the moment when the child faces for the first time linear equations with occurrences of the unknown on both sides of the equal sign". Our results show that most of our students were able to solve such equations without any prior instruction. But more importantly, the solution methods they used when the unknown was on both sides of the equal sign were the same as those used for equations where the unknown occurred twice on the same side. This leads us to view the demarcation between arithmetic and algebra in terms of a *cognitive gap*. This cognitive gap is characterized by *the student's inability to operate with or on the unknown*.

Our research was obviously influenced by the work of Kieran as well as that of Filloy and Rojano. That our results sometimes differ is due to various factors: different samples of students, different experimental conditions, different objectives, and a more systematic exploration of the students' informal solution procedures. In fact, although our study was not a teaching experiment, since no explicit instruction was involved, there is no question that the sequence of equations to be solved induced some learning. But this even strengthens the fact that our subjects could not operate with or on the unknown.

The identification of a cognitive gap between arithmetic and algebra has many interesting pedagogical implications. It defines more clearly a field of inquiry, pre-algebra, involving those intuitive algebraic ideas stemming from the presence of an unknown in a first degree equation, and focuses on the students' spontaneous solution procedures. But it also brings to light the need to more carefully develop and expand the arithmetic notions that will prove to be essential in a later course in algebra.

The need to expand the meaning of the equal sign has further been confirmed by our results, especially in its use to indicate the idea of decomposition, with the possibility that this might induce a student to read an equation from right to left. Our usual way of teaching the convention for the order of operations may result in some over generalizations by several students who may decide that addition takes precedence over subtraction in a string of such operations. Even if such problems occurred with only a few of our students, it must be remembered that we worked with a fairly gifted group and that these problems may be more common among average and weaker pupils. On the other hand, the detachment of a number from the preceding minus sign had a high incidence and this indicates that evaluating strings of operations is not a trivial problem. These difficulties indicate that some of the problems in early algebra find their origin in the students' arithmetic background and warrant further investigation.

It may be difficult for a teacher to appreciate the problems experienced by weaker or average students at an arithmetic level. But perhaps these are the difficulties that become major cognitive obstacles in their learning of algebra. Clearly, more research is needed to determine their prevalence and to find pedagogical interventions that might help the learner overcome them. By expanding the cognitive processes needed to bridge the gap between arithmetic and algebra many more students are likely to be recuperated. Perhaps even those students who are presently shunted away from any algebraic experience might benefit from an appropriate exposure to pre-algebra.

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