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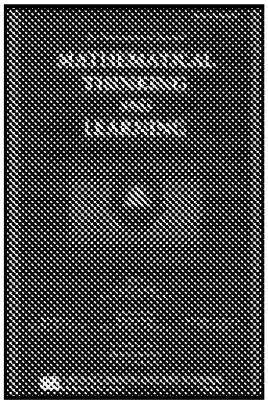

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A Longitudinal Examination of Middle School Students' Understanding of the Equal Sign and Equivalent Equations

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MATHEMATICAL THINKING AND LEARNING, *9*(3), 221–247
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A Longitudinal Examination of Middle School Students' Understanding of the Equal Sign and Equivalent Equations

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This longitudinal study investigated (a) middle school students' understanding of the equal sign, (b) students' performance solving equivalent equations problems, and (c) changes in students' understanding and performance over time. Written assessment data were collected from 81 students at four time points over a 3-year period. At the group level, understanding and performance improved over the middle school years. However, such improvements were gradual, with many students still showing weak understanding and poor performance at the end of grade 8. More sophisticated understanding of the equal sign was associated with better performance on equivalent equations problems. At the individual level, students displayed a variety of trajectories over the middle school years in their understanding of the equal sign and in their performance on equivalent equations problems. Further, students' performance on the equivalent equations problems varied as a function of when they acquired a sophisticated understanding of the equal sign. Those who acquired a relational understanding earlier were more successful at solving the equivalent equations problems at the end of grade 8.

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Algebra continues to receive substantial attention in the mathematics education community, particularly as questions concerning its nature and role in school mathematics are (re)considered. Traditionally conceived as the domain of an eighth- or ninth-grade mathematics course, recent reform efforts (e.g., Lacampagne, Blair, & Kaput, 1995; National Council of Teachers of Mathematics [NCTM], 2000; National Research Council, 1998) as well as research efforts (e.g., Bednarz, Kieran, & Lee, 1996; Carpenter, Franke, & Levi, 2003; Kaput, Carraher, & Blanton, 2007; RAND Mathematics Study Panel, 2003) have led to a reconceptualization of algebra as a continuous K–12 strand. Instantiating this conceptualization of algebra as a continuous K–12 strand requires bringing out the algebraic character of early mathematics activities (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006), and coherently connecting fundamental concepts of algebra across the elementary and secondary school years (Kaput, 1998).

One concept that is both fundamental to algebra understanding and ubiquitous in school mathematics at all levels is that of equality and, in particular, the equal sign. Developing an understanding of the equal sign has typically been considered mathematically straightforward—after its initial introduction during students' early elementary school education, little, if any, instructional time is explicitly spent on the equal sign in later grades. Yet, research suggests that many students at all grade levels have not developed an adequate understanding of the meaning of the equal sign. The results of such research highlight that "the notion of 'equal' is complex and difficult for students to comprehend" (RAND, 2003, p. 53) and, moreover, that it should be developed throughout the curriculum (NCTM, 2000). In this article, we report results from a longitudinal study that examined middle school students' understanding of the equal sign, the relationship between their understanding and their performance on problems using the equal sign, and the nature of changes in their understanding and performance over the course of their middle school years. The study is part of a larger, 5-year effort that seeks to understand and cultivate the development of middle school students' algebraic reasoning (Nathan & Koellner, this issue; see also http://labweb.education.wisc.edu/knuth/taar/).

RESEARCH ON STUDENT UNDERSTANDING OF THE EQUAL SIGN AND PERFORMANCE SOLVING EQUIVALENT EQUATIONS PROBLEMS

Student understanding of the equal sign has received considerable research attention in recent years (e.g., Alibali, 1999; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; McNeil & Alibali, 2005). A primary result emerging from these studies is that

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many students view the equal sign operationally, that is, as a "do something signal" that announces the result of an arithmetic operation. For example, Falkner et al. (1999) found that a substantial number of first- through sixth-grade students' solutions to the equation $8 + 4 = \square + 5$ included 12, 17, or 12 and 17—solutions that are consistent with a view of the equal sign as announcing the result of an arithmetic operation. Although the "operational" view may suffice when solving typical elementary school arithmetic problems (e.g., $3 + 5 = \square$), it can be problematic when students encounter more complex equations in later grades (e.g., $3x + 5 = 11$, $2x - 3 = 4x + 5$). In the latter case, research suggests that learning to operate on the structure of such equations (i.e., perform the same operations on both sides) may be easier for students who view equations as objects with symmetric balance (Kieran, 1992). Developing a view of the symmetric nature of an equation requires understanding that the equal sign is a mathematical symbol representing a relationship between quantities rather than a signal to perform arithmetic operations.

Researchers have posited that students' difficulties in understanding the equivalence among transformed equations in an equation-solving chain may be attributed, in part, to their not apprehending the equivalence relationship between the left- and right-hand expressions in algebraic equations (Kieran, 1989). Steinberg, Sleeman, and Ktorza (1990), for example, found that although many eighth- and ninth-grade students could perform appropriate transformations when solving equations, many could not determine whether two given equations were equivalent, particularly in cases in which nonstandard transformations had been applied to the first equation (e.g., $2x + 6 = 10$ » $2x + 6 - 8 = 10 - 8$), as such cases require recognition that the transformation preserves the equivalence relation expressed in the first equation. Similarly, results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress (NCTM, 1989) indicate that many pre-algebra, first-year algebra, and second-year algebra students had difficulty with equivalent equations items: When asked to identify a pair of equations that were equivalent, only 30% of the pre-algebra students, 50% of the first-year algebra students, and 60% of the second-year algebra students responded correctly.

In considering student performance on equivalent equations problems, researchers have suggested that many students' difficulties may be attributed to their misconceptions about the meaning of the equal sign (Carpenter et al., 2003; Kieran, 1981). Thus, one goal of this article is to provide evidence that understanding the equal sign is related to performance solving equivalent equations problems. A second goal is to present results concerning the nature of changes in student understanding and performance as they progress through middle school. Finally, a third goal is to identify plausible learning trajectories for equal sign understanding and to connect these trajectories to student performance solving equivalent equations problems. The research presented in this article is guided by the following questions: What meanings do middle school students ascribe to the equal sign, and how do these change over time? What is the relationship between the meanings ascribed to the equal sign and performance on problems using the equal sign? How is performance on problems using the equal sign related to *when* students acquire a more sophisticated understanding of the equal sign?

METHOD

Participants

Participants were 81 middle school students drawn from an ethnically diverse middle school in the American Midwest. At the time of the study, the demographic breakdown of the school's student population was as follows: 62% White, 25% African American, 7% Asian, and 5% Hispanic. The curricular program utilized by the middle school was *Connected Mathematics*, and of relevance to this study, the curricular development of algebra begins in the grade 6 units, with increasing attention given to algebra in the grade 7 and grade 8 units (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). In addition, traditional algebraic topics such as solving linear equations receive explicit attention beginning in grade 7.

Data Collection

Longitudinal data were collected at four time points over a three-year period: beginning of grade 6, beginning of grade 7, beginning of grade 8, and end of grade 8. The data that are the focus of this article consist of students' responses to a subset of items from a written assessment (administered at each of the four time points) that targeted various aspects of algebra, including understanding of the equal sign and of variables, and fluency with different representations of mathematical information (e.g., equations, tables, graphs). Our focus in this article is on three items designed to elicit students' understanding of the equal sign (*equal sign item*) and their performance on problems that (potentially) required use of their equal sign understanding (*equivalent equations items*). Students' responses to the representational fluency items are discussed in another article in this special issue (Nathan & Kim, this issue); students' responses to the variable items will be discussed in a forthcoming article.

There were three forms of the assessment, each containing a slightly different subset of items. We used multiple forms so that we could gather data on a larger set of items than we could with a single form. Forms were randomly assigned to students within each grade. Every student responded to the equal sign item (i.e., it was presented on all three forms), 55 students (roughly two-thirds) responded to one of the equivalent equations items (i.e., it was presented on two of the three forms), and 26 students (roughly one-third) responded to the other equivalent equations

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The following questions are about this statement:

3 + 4 = 7
↑

a) The arrow above points to a symbol. What is the name of the symbol?

b) What does the symbol mean?

c) Can the symbol mean anything else? If yes, please explain.

FIGURE 1 Interpreting the equal sign (Item 1).

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item (i.e., it was presented on one of the three forms). Thus, every student received the equal sign item and one of the two equivalent equations items.

The equal sign item (see Figure 1) required students to (a) name the equal sign symbol (first prompt), (b) provide a statement regarding the symbol's meaning (second prompt), and (c) if possible, provide a statement regarding an alternative meaning (third prompt). The rationale for the first prompt was to preempt students from using the name of the symbol in their response to the second prompt (e.g., "the symbol means equal"). The rationale for the third prompt was based on our previous work (e. g., Rittle-Johnson & Alibali, 1999), in which we found that students often offer more than one interpretation when given the opportunity. This measure of equal sign understanding was based on the seminal work of Behr et al. (1980) and Baroody and Ginsburg (1983), and it has been used extensively in previous research. The measure has good convergent validity; students' responses correlate with their responses on another measure of equal sign understanding (McNeil & Alibali, 2005). It also has good concurrent validity; students who have more math experience perform better on the measure than students who have less math experience (McNeil & Alibali, 2005), and students who perform better are more likely to solve algebraic equations correctly, even when controlling for general mathematics ability (Knuth, Stephens, McNeil, & Alibali, 2006).

The equivalent equations items (see Figures 2 and 3) essentially required students to determine whether the missing value in one equation was the same as the missing value in a second, equivalent equation. The design of Item 2 (see Figure 2) was based on items used in previous research (Steinberg et al., 1990).

Is the value of *n* the same in the following two equations? *Explain your reasoning*.

$2 \times n + 15 = 31$ $\qquad\qquad$ $2 \times n + 15 - 9 = 31 - 9$

Is the number that goes in the $\square$ the same number in the following two equations? *Explain your reasoning*.

$2 \times \square + 15 = 31$ $\qquad\qquad$ $2 \times \square + 15 - 9 = 31 - 9$

FIGURE 2 Equivalent equations item (Item 2). Students received either the literal symbol version of the item or the box version.

In the equation $\square + 18 = 35$, the number that goes in the $\square$ is 17.

Can you use this fact to help you figure out what number goes in the

$\square$ in the equation $\square + 18 + 27 = 35 + 27$?

***Explain your reasoning*.**

FIGURE 3 Equivalent equations item (Item 3).

It is worth noting that the transformation applied in the second (equivalent) equation in Item 2 is atypical in that it is not a transformation that would be applied in the process of solving the first equation (i.e., subtracting 15 from both sides of the equation). The rationale for the atypical transformation was to deter students from simply recalling procedures used in solving an equation and thus potentially relying on knowing the procedure to respond that the solutions to the two equations are the same. Similar to Item 2, Item 3 (see Figure 3) also included an atypical transformation; however, the solution to the first equation in the pair was provided for students. The rationale for including the solution was to remove the "need" to solve the first equation and, thus, potentially focus students' thinking on the nature of the second equation in relation to the first equation. For both items, we expected that students who viewed the equal sign as representing a relationship between quantities would recognize that the transformation performed on the second equation of each pair preserved the quantitative relationship expressed in the first equation of each pair, and thus conclude that the solution to each member of the pair of equations is the same.

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For Item 3 (which appeared on one of the assessment forms), we used a box in each equation rather than the more prototypical (in terms of algebra) literal symbol (e.g., *n*), thinking that the former would likely be more familiar to students based on its frequent use in elementary school (e.g., "What value will make the number sentence, 8 + □ = 12, true?"). For Item 2 (which appeared on two of the assessment forms), we used a □ on one form and a literal symbol (*n*) on the other. Student performance did not differ significantly across versions for Item 2; therefore, the analyses reported in this article collapse across these two versions.

Coding

For all three items, responses that students left blank or for which they wrote "I don't know" were grouped into a *no response/don't know* category, and responses that were not sufficiently frequent to warrant their own codes were grouped into an *other* category.

Equal Sign Item Coding. Student responses to parts (b) and (c) of the equal sign item (Item 1) were coded as *relational*, *operational*, *unspecified equal*, or *other*, with the majority of responses falling into the first two categories. A response was coded as *relational* if a student expressed the general idea that the equal sign means "the same as," as *operational* if a student expressed the general idea that the equal sign means "add the numbers" or "the answer," and as *unspecified equal* if a student provided a definition using the words "equal" or "equals" but did not provide enough additional information to suggest a more specific understanding. The *other* category included definitions that did not address the mathematical meaning of the symbol. Representative examples for each category are presented in Table 1. This coding system has been used in past

TABLE 1
Coding Categories and Representative Examples for the Equal Sign Definition Item

Category	*Example*
Relational	"It means that both sides of the equation are equal." (7th)
	"The numbers on either side are balanced." (8th)
Operational	"After the symbol it shows the answer." (6th)
	"It means to add every thing up." (7th)
Unspecified equal	"It means something that equals something else." (6th)
	"It is equal to." (8th)
Other	"It could mean smiley. =) " (6th)
	"On a calculator it can mean enter." (7th)
	"A mother has to treat her kids equal." (8th)

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work by Knuth et al. (2006), McNeil and Alibali (2005), and Rittle-Johnson and Alibali (1999), among others.

In addition to coding responses to parts (b) and (c) separately, students were also assigned an overall code indicating their "best" interpretation. Many students provided two interpretations, sometimes one *relational* and one *operational*; in such cases, the responses were assigned an overall code of *relational* to reflect the students' optimal level of reasoning. At each time point, between 31% and 65% of students who provided a relational definition also provided a nonrelational definition.

Equivalent Equations Items Coding. For the first equivalent equations item (Item 2), we coded both correctness and the strategy used by each student. For the second equivalent equations item (Item 3) we coded only the strategy used by each student. Responses were coded as *correct* if the student responded that the number that goes in the box is the same in each equation (irrespective of strategy). Students' strategies were classified into one of the following six categories: *answer after equal sign*, *solve and compare*, *recognize equivalence*, *substitution*, *other*, and *no response/don't know*. Responses coded as *answer after equal sign* were responses in which a student indicated that the "answer" is the value of the right side of the equation (e.g., for Item 2, "No, the answer to the second equation is 22 and the answer to the first is 31") or the value of the first number to the right of the equal sign (e.g., for Item 2, "Yes, the answer to both equations is 31"). Re-

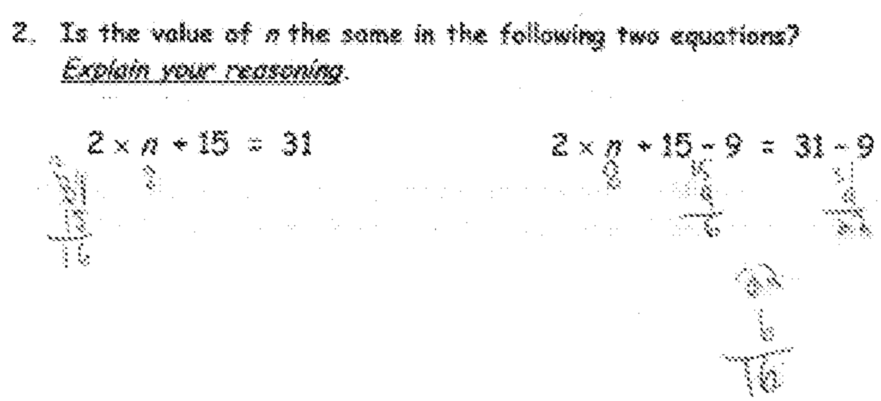
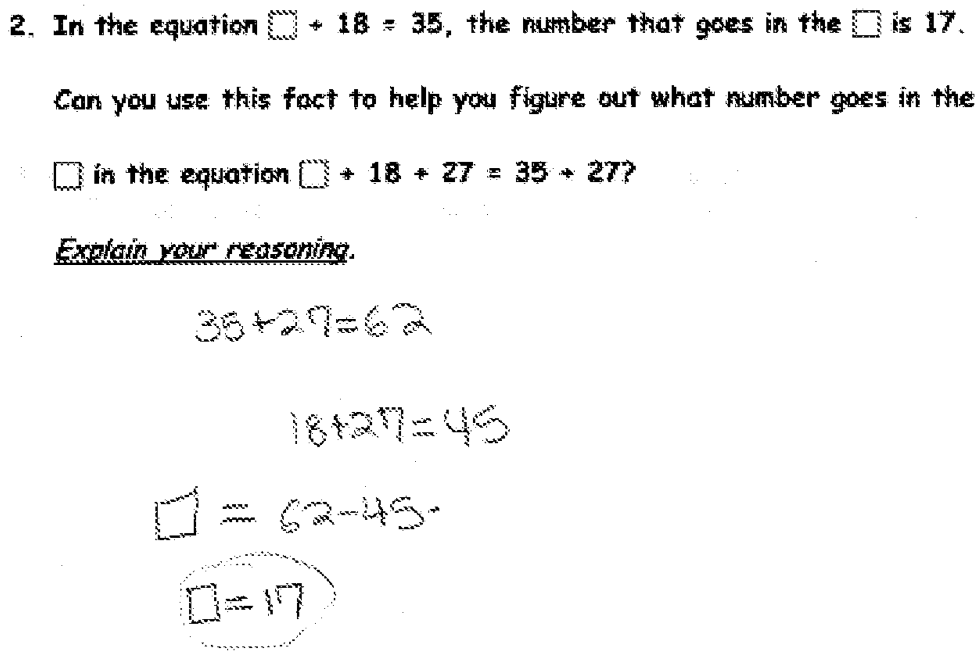


FIGURE 4 Solve and compare strategy (Item 2).


2. In the equation □ + 18 = 35, the number that goes in the □ is 17.

Can you use this fact to help you figure out what number goes in the □ in the equation □ + 18 + 27 = 35 + 27?

Explain your reasoning.

35+27=62

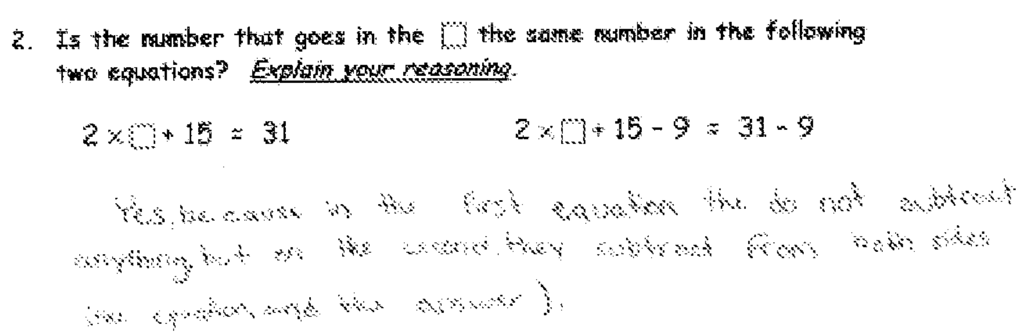
18+27=45

□ = 62-45

□=17

FIGURE 5 Solve and compare strategy (Item 3).

sponses coded as *solve and compare* were responses in which a student: (a) determined the solution to each equation separately and then compared the solutions (see, for example, Figure 4), (b) determined the solution to one equation and then checked it by substitution to see if that value satisfied the second equation, or (c) determined the solution to the second equation and then compared the solution to the given solution (see, for example, Figure 5). Responses coded as *recognize equivalence* were responses in which a student recognized that the transformation performed on the second equation preserved the equivalence relation expressed in the first equation (see, for example, Figures 6 and 7). Finally, responses coded as


2. Is the number that goes in the □ the same number in the following two equations? Explain your reasoning.

2 × □ + 15 = 31 2 × □ + 15 − 9 = 31 − 9

Yes, because in the first equation the do not subtract anything but on the second they subtract from both sides (the equation and the answer).

FIGURE 6 Recognize equivalence strategy (Item 2).

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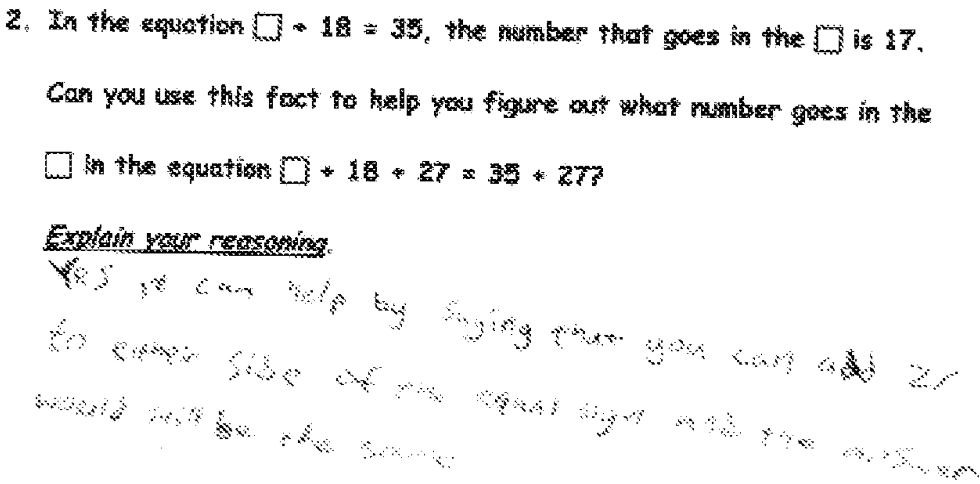
2. In the equation $\square + 18 = 35$, the number that goes in the $\square$ is 17.

Can you use this fact to help you figure out what number goes in the $\square$ in the equation $\square + 18 + 27 = 35 + 27$?

Explain your reasoning.

FIGURE 7 Recognize equivalence strategy (Item 3).

substitution were responses (to Item 3) in which a student substituted the value of 17 into the second equation to check whether the two equations had the same solution. Knuth, Alibali, McNeil, Weinberg, and Stephens (2005) used this coding system in previous research.

Reliability of Coding Procedures. To assess reliability of the coding procedures, a second coder rescored approximately 20% of the data. Agreement between coders was 96% for coding students' interpretations of the equal sign, 95% for coding correctness of students' responses to the first equivalent equations item (Item 2), and 95% for coding students' strategies on both equivalent equations items (Items 2 and 3).

RESULTS AND DISCUSSION

The results will be presented in two major sections. The first section presents analyses of performance and change over time for the group of students as a whole. In this section, we consider changes over time in students' definitions of the equal sign, changes over time in performance on the equivalent equations items, and links between students' equal sign definitions and their performance on the equivalent equations items. The second section presents analyses with the individual student as the unit of analysis. In this section, we examine whether students continue to display sophisticated reasoning at later time points, once they have done so at an earlier time point. We also examine whether students tend to display a relational definition of the equal sign *before*

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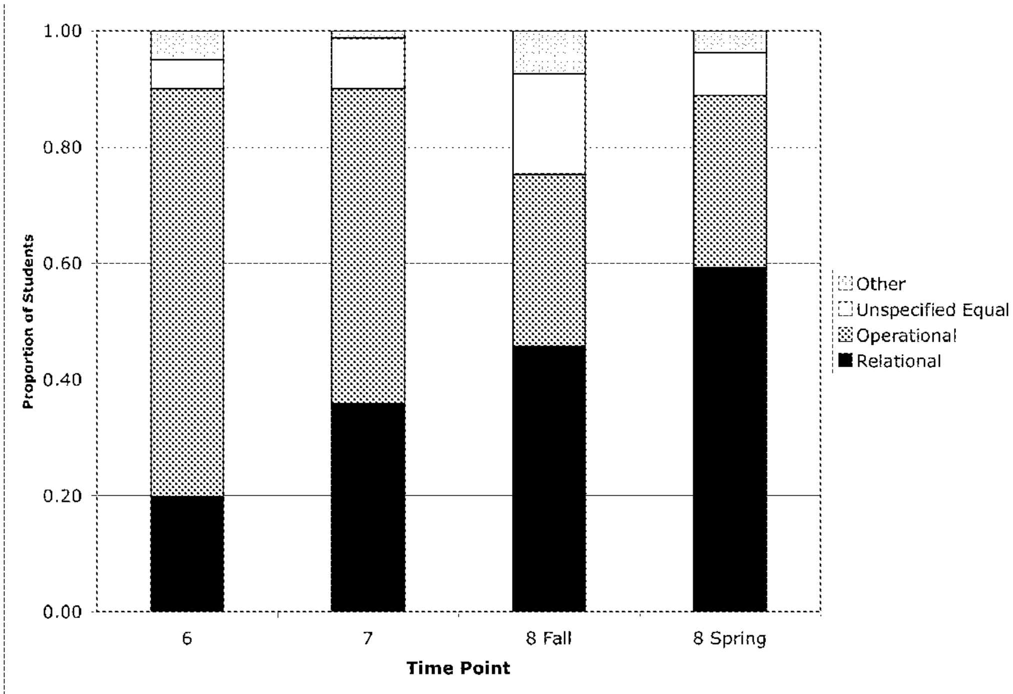

FIGURE 8 Proportion of students who offered each type of equal sign definition at each time point.

or *after* beginning to use the sophisticated, recognize-equivalence strategy. Finally, we consider whether students' performance on the equivalent equations items varies as a function of *when* they acquire a relational understanding of the equal sign.

PERFORMANCE AND CHANGE OVER TIME: GROUP LEVEL ANALYSES

Change in Performance Over Time

We first examined whether the definitions that students offered for the equal sign changed over the course of the study. As seen in Figure 8, the proportion of students who provided a relational definition as their best definition increased across the four time points. Only 20% of students offered such a definition at grade 6, but nearly 60% of students did so by spring of grade 8.

A similar improvement across time points was seen in students' success at solving the equivalent equations items. Among students who received Item 2 ($N = 55$), the proportion of students who correctly stated that both equations have the same solution increased across the four time points, with only about half of the students doing so at grade 6, but over 75% of students doing so by spring of grade 8 (Figure 9).

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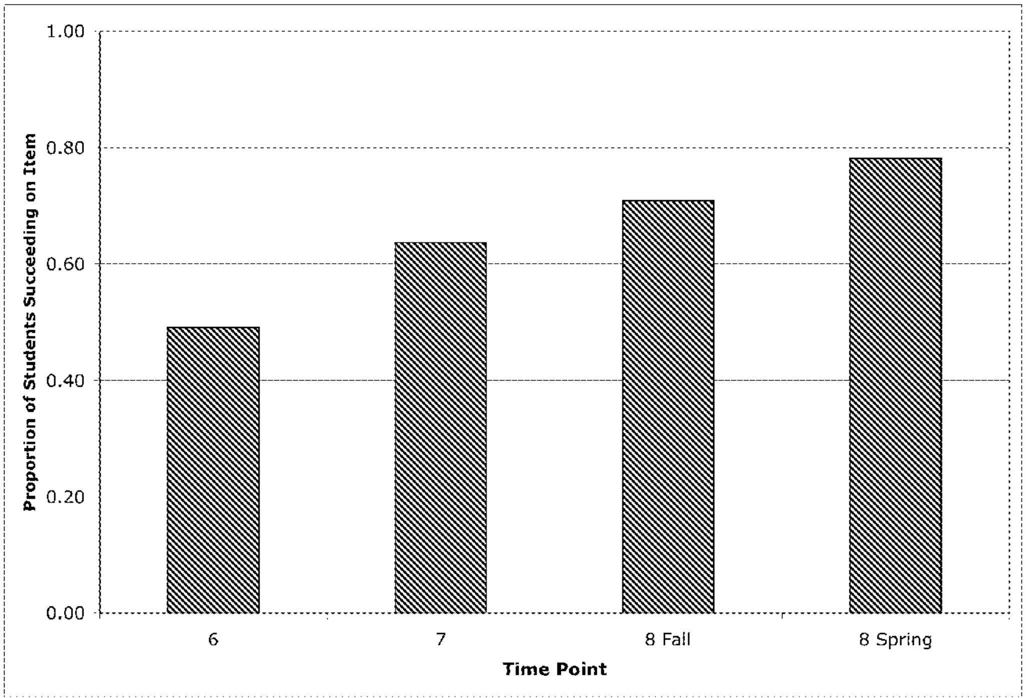

FIGURE 9 Proportion of students who correctly stated that the two equations have the same solutions on the equivalent equations item (Item 2).

Students' strategies for solving the equivalent equations items changed over time as well. Among students who received Item 2 ($N = 55$), the proportion of students who applied the recognize-equivalence strategy increased across grades, from 9% to 40%, as seen in Figure 10. Likewise, among students who received Item 3 ($N = 26$), the proportion of students who used this strategy increased from 23% to 42% (Figure 10). Given that patterns of change over time in use of the recognize-equivalence strategy were comparable for Items 2 and 3, we collapsed across versions in the remainder of the analyses.

Taken together, these patterns suggest that, at the group level, students' understanding of the equal sign and their performance using their understanding showed substantial improvement across the middle grades. Many students, however, still did not exhibit a relational understanding of the equal sign and still performed poorly on the equivalent equations items, even in the spring of grade 8.

Links Between Equal Sign Definitions and Performance on Equivalent Equations Items

The results thus far show that performance on both the equal sign definition item and the equivalent equations items improved across the grade levels. However,

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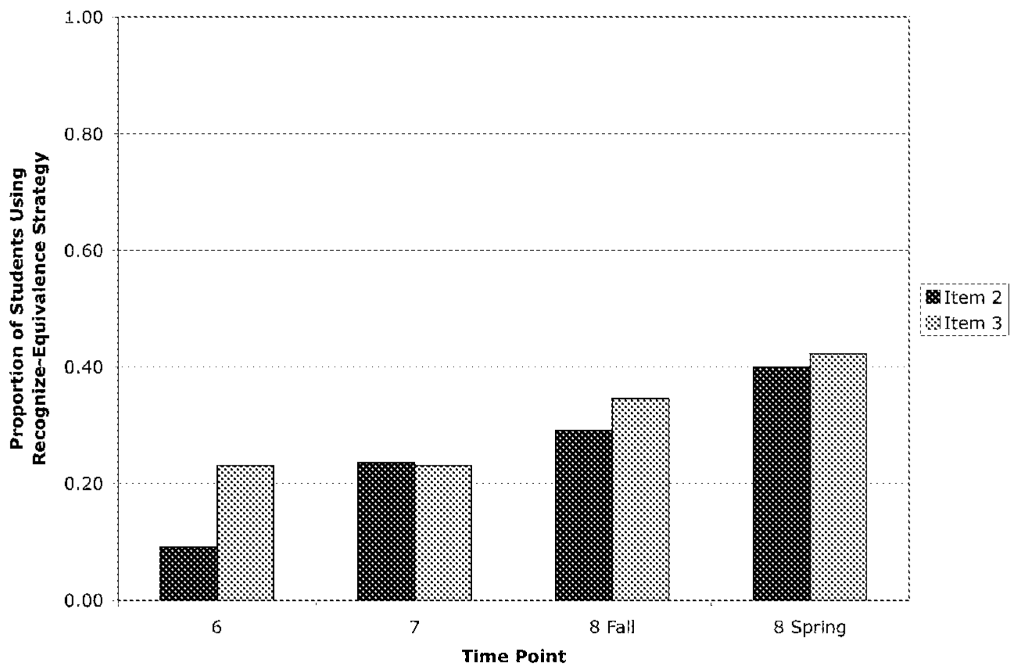

FIGURE 10 Proportion of students who used the recognize-equivalence strategy on the equivalent equations items (Items 2 and 3). (Note: the performance difference across items at Grade 6 is not significant.)

they do not reveal whether the same students tended to perform well on both items. This was, in fact, the case. At each time point, students who demonstrated a relational understanding of the equal sign were (a) more likely to state that the two equations were equivalent (on Item 2) (Figure 11), and (b) more likely to use the sophisticated, recognize-equivalence strategy (on both items) (Figure 12). The association between relational understanding of the equal sign and stating that the two equations were equivalent (Item 2) was statistically significant at grade 7, χ^2 $(1, N = 55) = 8.37$, $p = .004$, and approached significance at grade 8 (spring), χ^2 $(1, N = 55) = 3.20$, $p = .07$. The association between relational understanding of the equal sign and using the recognize-equivalence strategy was statistically significant at grade 7, χ^2 $(1, N = 81) = 8.08$, $p = .004$, and at grade 8 (fall), χ^2 $(2, N = 81) = 7.26$, $p = .007$. Although the links were not statistically significant at every time point separately, the overall patterns are clear. These results replicate previous findings (Knuth et al., 2005), and they converge with prior research showing that students who have a relational understanding of the equal sign perform better at solving linear equations than students who do not have a relational understanding (Knuth et al., 2006).

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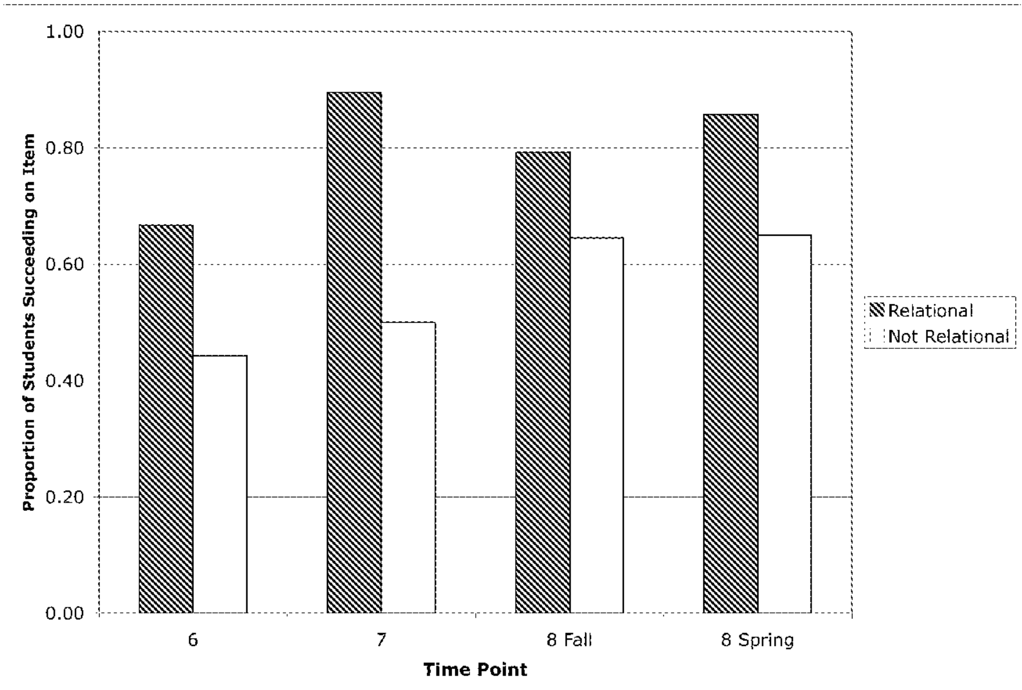

FIGURE 11 Proportion of students who correctly stated that the two equations have the same solutions on the equivalent equations item (Item 2), as a function of the type of definition they offered for the equal sign.

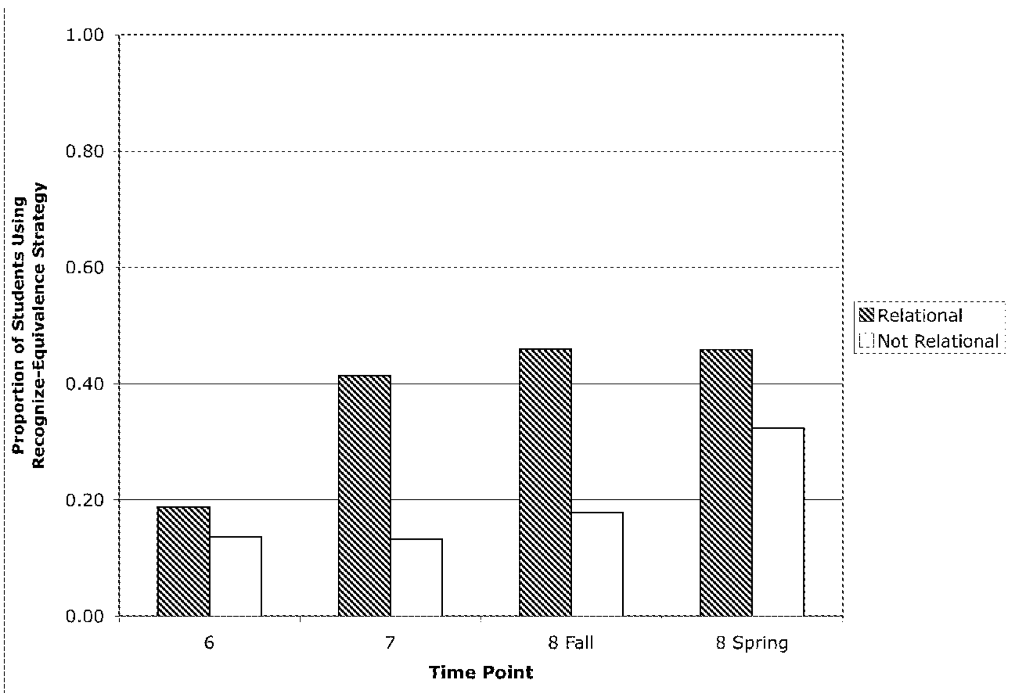

FIGURE 12 Proportion of students who used the recognize-equivalence strategy on the equivalent equations item (collapsed across Items 2 & 3), as a function of the type of definition they offered for the equal sign.

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Discussion of Group Level Analyses

The present findings are consistent with past research indicating that many middle school students hold an operational view of the equal sign. We observed improvement in students' understanding across the middle grades, but this improvement appears to be gradual, both for understanding of the equal sign and for recognizing the equivalence of equations, with many students still showing weak understanding at the end of grade 8.

The rather gradual improvement across grades is perhaps unsurprising given the lack of explicit focus on equality in middle school mathematics curricula, including the one used by students in this study, *Connected Mathematics*. Many middle school teachers believe that most middle school students have a good grasp of the meaning of the equal sign (Asquith, Stephens, Knuth, & Alibali, this issue), in part because they believe that the equal sign was an explicit focus of instruction at the elementary level. Furthermore, middle school curricular materials often present the equal sign in contexts that support an operational interpretation, such as the "operations equals answer" problem structure (e.g., $38 + 27 = \square$), which is compatible with the interpretation that the equal sign means "find the total" or "put the answer" (McNeil et al., 2006). These factors presumably contribute to students maintaining unsophisticated interpretations of the equal sign, even through the end of middle school.

Our findings replicate past work suggesting that it matters whether students have a relational understanding of the equal sign. Students who displayed a relational understanding of the equal sign performed better on the equivalent equations items than students who did not display such understanding, both in correctly judging that two equations were equivalent (Item 2) and in using the sophisticated, recognize-equivalence strategy (Items 2 and 3). Kieran (1992) has argued that "one of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric and transitive character of equality—sometimes referred to as the 'left-right equivalence' of the equal sign" (p. 398). A relational view of the equal sign allows students to interpret equations appropriately, and appropriate interpretations can guide judgments about the equivalence of equations.

PERFORMANCE AND CHANGE OVER TIME: INDIVIDUAL LEVEL ANALYSES

The longitudinal design of this study allows us to address questions, not only about group-level patterns of performance and change over time, but also about individual students' trajectories. We turn next to analyses of performance and change over time that utilize the individual student as the unit of analysis.

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Change in Performance Over Time

Once students offered a relational definition, did they continue to do so at later time points? Table 2 presents the proportion of students who displayed various patterns in their equal sign definitions across the four time points. As an example, the following responses are from a student who provided operational definitions in grades 6 and 7 and relational definitions in both fall and spring of grade 8:

> "The answer comes next." (grade 6)
>
> "It means that the next number is the answer." (grade 7)
>
> "It means that the numbers before it are equal to the numbers after it." (fall, grade 8)
>
> "It means 'is the same as.'" (spring, grade 8)

Many students continued to offer relational definitions once they started doing so—15% did so at all four time points, 19% became relational at grade 7 and remained so for the remaining two time points, and 12% became relational in the fall of grade 8 and remained so at the final time point, for a total of 46% of students. Only 12% of students "vacillated" between relational and operational definitions, in the sense that they offered an operational definition at least once after having offered a relational definition at a prior time point.

We also examined whether most students continued to use the recognize-equivalence strategy after they did so at least once. Table 3 presents the proportion of students who displayed various patterns in their strategy use for the equivalent equations items across the four time points (collapsed across Items 2 and 3). Some students continued to use the recognize-equivalence

TABLE 2
Percent of Students Who Displayed Each Type of Trajectory Across the Four Time Points in Their Equal Sign Definitions

Pattern	*Percent of Students*
Operational at all 4 time points	25
Relational at all 4 time points	15
Become relational at grade 7 and remain so	19
Become relational at grade 8 (Fall) and remain so	12
Become relational at grade 8 (Spring)	14
Vacillate	12
Other	4

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TABLE 3
Percent of Students Who Displayed Each Type of Trajectory Across the Four Time Points in Their Use of the Recognize-Equivalence Strategy on the Equivalent Equations Items

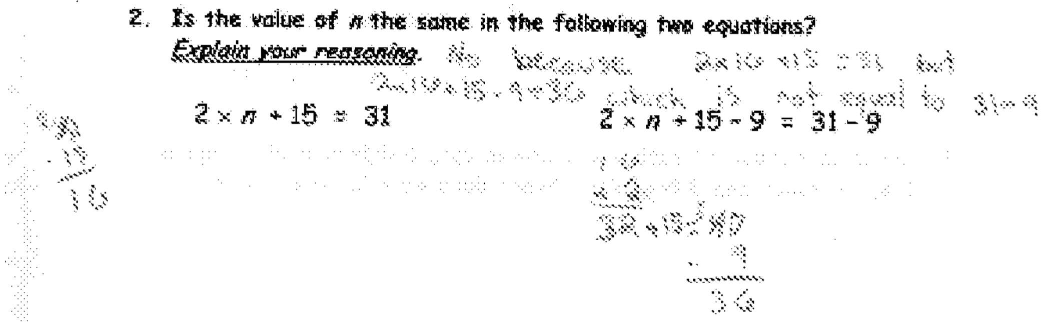
Pattern	*Percent of Students*
Never use recognize-equivalence	47
Use recognize-equivalence at all 4 time points	5
Begin to use recognize-equivalence at grade 7 and continue to do so	7
Begin to use recognize-equivalence at grade 8 (Fall) and continue to do so	5
Begin to use recognize-equivalence at grade 8 (Spring)	12
Vacillate	23

strategy once they started doing so—5% used it at all four time points, 7% began to use it in grade 7 and continued to do so at the remaining two time points, and 5% began to use it in the fall of grade 8 and continued to do so at the final time point, for a total of 17% of students. The responses in Figure 13, for example, are from a student who, in grade 6, used the *solve-and-compare* strategy (arriving at an incorrect answer!) and then used the *recognize-equivalence* strategy thereafter. In contrast, 23% of students "vacillated" between the recognize-equivalence strategy and other strategies, in the sense that they used a less sophisticated strategy at least once after having used the recognize-equivalence strategy at an earlier time point. For example, the responses in Figure 14 are from a student who used the *recognize-equivalence* strategy in grade 6, did not use any overt strategy in grade 7, used the *solve and compare* strategy in the fall of grade 8, and used the *recognize equivalence* strategy in the spring of grade 8.

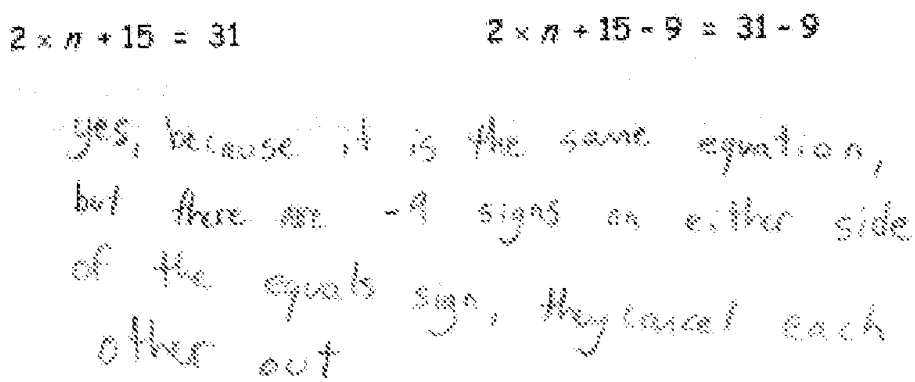
Which Type of Knowledge Comes First—Relational Definitions or the Recognize-Equivalence Strategy?

The longitudinal design of our study also allows us to examine which type of knowledge comes first—the ability to provide an explicit, relational definition of the equal sign, or the ability to use the recognize-equivalence strategy. Put another way, what "leads" in acquiring knowledge in this general area—explicit conceptual knowledge of the meaning of the equal sign, or the ability to apply the (perhaps implicit) idea of equivalence in solving a problem?

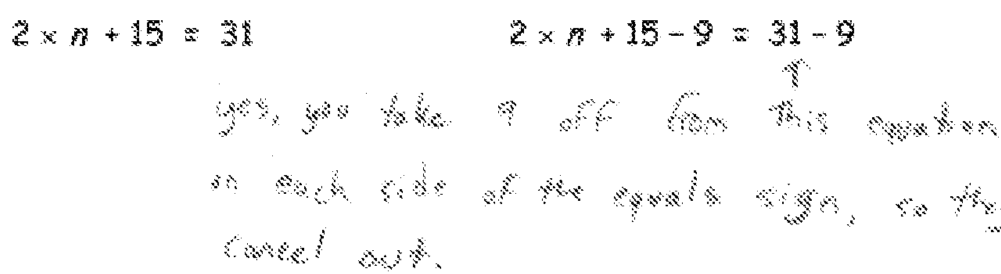
Of the 81 students in the sample, 17 students (21%) neither produced a relational definition of the equal sign nor used the recognize-equivalence strategy to solve the equivalent equations items during the study, and 13 students (16%) began to do both at the same time point. Of the remaining 51 students, 16 (31%, or 18% of the total sample) used the recognize-equivalence strategy at least one time point



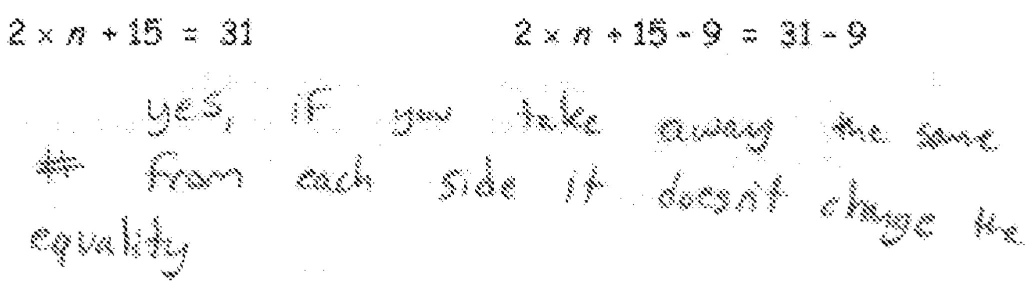
(a) Grade 6: *Solve and compare*



(b) Grade 7: *Recognize equivalence*



(c) Grade 8 Fall: *Recognize equivalence*



(d) Grade 8 Spring: *Recognize equivalence*

FIGURE 13 Student work showing initial use of the *solve and compare* strategy (at Grade 6) and then the *recognize equivalence* strategy thereafter.

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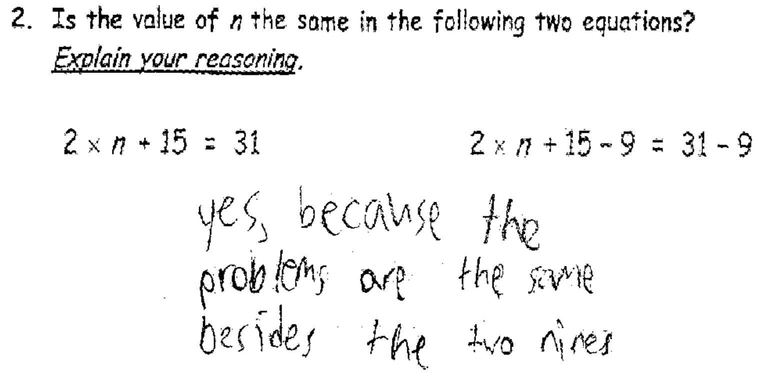
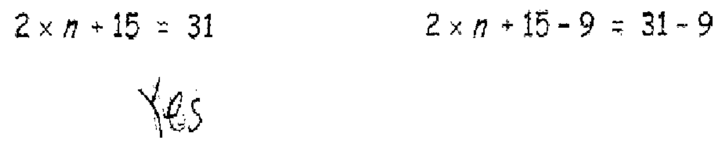
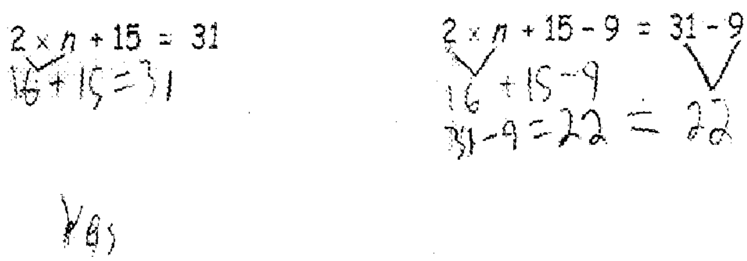


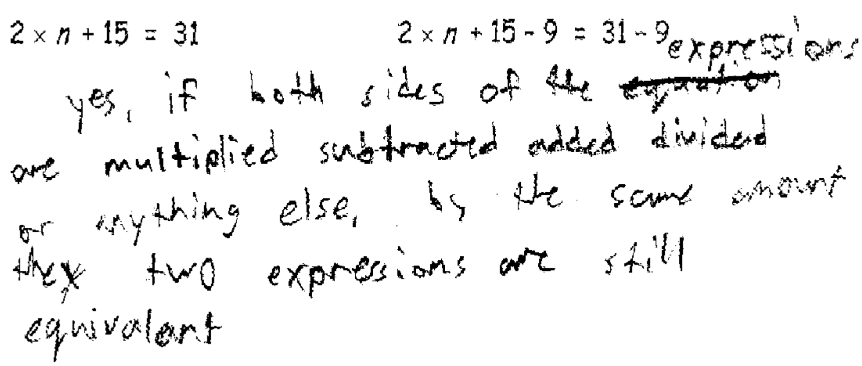
(a) Grade 6: *Recognize equivalence*



(b) Grade 7: *No strategy*



(c) Grade 8 Fall: *Solve and compare*



(d) Grade 8 Spring: *Recognize equivalence*

FIGURE 14. Student work showing vacillation between the *solve and compare* strategy and the *recognize equivalence* strategy.

before they offered a relational definition of the equal sign, and 35 (69%, or 43% of the total sample) offered a relational definition of the equal sign at least one time point before they used the recognize-equivalence strategy. If there were no systematic ordering of the two achievements, students who show one achievement before the other should be evenly split between those who offer a relational definition first and those who use the recognize-equivalence strategy first. Assuming a binomial distribution with p = .50 and 51 "trials," the probability of having 35 "trials" or more in one category is p = .005. Thus, the data suggest that the ability to offer a relational definition generally precedes the ability to recognize the equivalence of two equations.

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Does It Matter When Students Acquire an Understanding of the Equal Sign?

It is possible that it may not matter *when* students acquire a relational understanding of the equal sign, as long as they do so eventually. To test this idea, we examined whether students' performance on the equivalent equations items varied as a function of the *timing* of their acquisition of a relational understanding of the equal sign. Our analysis focused on the subset of students who did display a relational understanding of the equal sign during the study, and we examined the likelihood that students solved the equivalent equations item successfully at the final time point (grade 8, spring) as a function of *when* they first displayed that understanding. The data are presented in Figure 15.

For statistical analysis, we collapsed the students into three groups: those who offered a relational definition early in the study (in grade 6 or 7), those who offered a relational definition later in the study (in grade 8, fall or spring), and those who never offered a relational definition. The proportion of students who succeeded on the equivalent equations item (Item 2) varied significantly across groups, χ^2 (2, N = 55) = 7.33, p = .025. Not surprisingly, students who never displayed a relational understanding of the equal sign were least likely to succeed on the equivalent equations item (see Figure 15). More interesting is the fact that students who first displayed a relational understanding of the equal sign in grade 6 or grade 7 were more likely than students who first displayed a relational understanding of the equal sign in grade 8 (fall or spring) to succeed on the equivalent equations item in the spring of grade 8 (95% vs. 75%).

We also examined the likelihood that students used the recognize-equivalence strategy at the final time point (on both Items 2 and 3) as a function of when they first displayed a relational understanding of the equal sign. The data are presented in Figure 16. Again, for statistical analysis, we collapsed the students into three groups: those who offered a relational definition early in the study (in grade 6 or 7), those who offered a relational definition later in the study (in grade 8, fall or spring), and those who never offered a relational definition. The proportion of stu-

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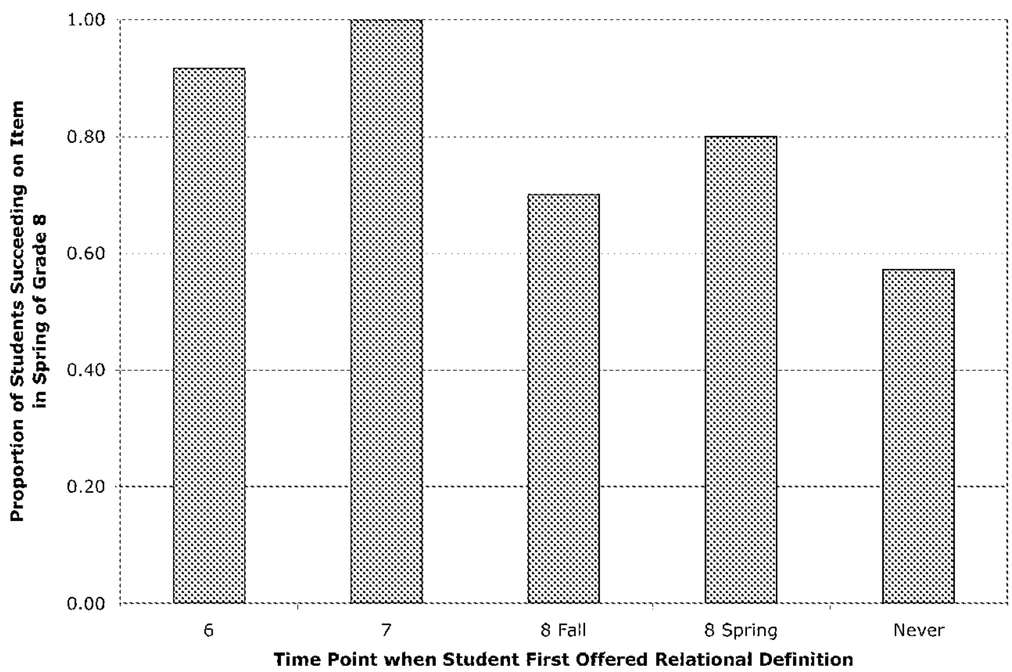

FIGURE 15 Proportion of students who correctly stated that the two equations have the same solutions on the equivalent equations item (Item 2), as a function of the time point at which they first offered a relational definition for the equal sign.

dents who used the recognize-equivalence strategy varied significantly across groups, χ^2 (2, $N = 81$) = 9.62, $p = .008$. As would be expected, students who never displayed a relational understanding of the equal sign were least likely to use the recognize-equivalence strategy (see Figure 16). More compelling, students who first displayed a relational understanding of the equal sign in grade 6 or grade 7 were nearly twice as likely as students who first displayed a relational understanding of the equal sign in grade 8 (fall or spring) to use the recognize-equivalence strategy in spring of grade 8 (61% vs. 32%), χ^2 (1, $N = 55$) = 4.66, $p = .03$.

Moreover, this pattern holds even if we limit the analysis to students who offered a relational definition of the equal sign at the final time point (i.e., if we exclude students who offered a relational definition at an earlier time point but then shifted back to an operational definition or an unspecified-equal definition at the final time point) (61% vs. 25%), χ^2 1, ($N = 48$) = 5.99, $p = .01$. The important point here is that these students all displayed the same level of knowledge about the equal sign symbol at the final time point, but their likelihood of using the sophisticated recognize-equivalence strategy at that time point depended on *when* they had acquired their knowledge of the equal sign.

Thus, students' performance varied as a function of when they had acquired a relational understanding of the equal sign. Individual students' developmental his-

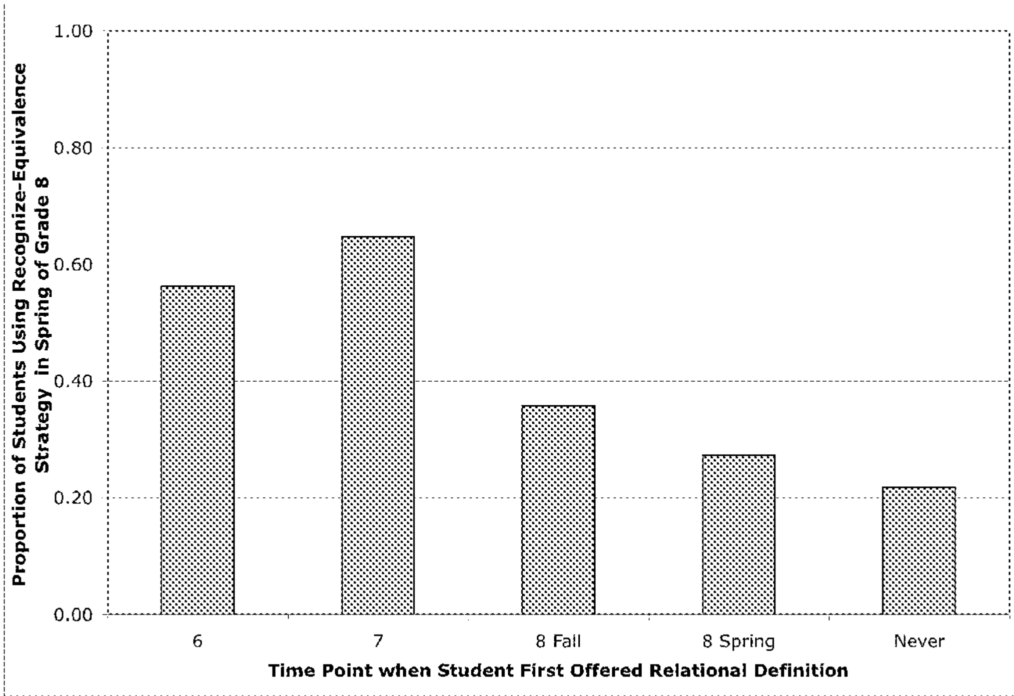

FIGURE 16 Proportion of students who used the recognize-equivalence strategy on the equivalent equations item (collapsed across Items 2 & 3), as a function of the time point at which they first offered a relational definition for the equal sign.

tories were related to performance on the equivalent equations item, as well as use of the recognize-equivalence strategy.

Discussion of Individual Level Analyses

Students displayed varying trajectories across the four time points in their understanding of the equal sign and in their performance on the equivalent equations items. It was not the case that once students displayed understanding, they always continued to do so. Some students who offered a relational definition for the equal sign at an early time point later offered an operational definition, and some students who used the recognize-equivalence strategy on the equivalent equations item at an early time point used less sophisticated strategies at later time points.

What might contribute to the variability we observed in whether students displayed their knowledge? One likely possibility is that students' early relational understandings are fragile, and students may continue to hold the operational interpretation as well as the relational one. In support of this view, past research has shown that the context in which equal sign definitions are elicited affects the likelihood that students offer a relational definition, particularly in the middle school years (McNeil & Alibali, 2005). In the present study, the context of students' math-

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ematics classrooms around the time of the assessments may have affected students' tendencies to offer a relational definition or to activate a relational interpretation when solving the equivalent equations items. If recent classroom activities were compatible with an operational view of the equal sign (e.g., arithmetic practice, with many items with an "operations equals answer" problem structure), students may have been more likely to activate an operational interpretation, whereas if recent classroom activities were compatible with a relational view of the equal sign (e.g., exercises about equivalent fractions), students may have been more likely to activate a relational interpretation.

It is worth noting that the assessment items we used are also a form of "context" that may affect students' propensity to activate relational or operational ideas about the equal sign. The items we used represent only three of a universe of possible items that we could have used to assess students' understanding, and other items may have elicited different types of responses from the very same students, at the very same time points.

We observed greater stability in students' definitions of the equal sign than in their use of the recognize-equivalence strategy on the equivalent equations problems. That is, students were more likely to consistently define the equal sign relationally than to consistently use the recognize-equivalence strategy. Why might this be the case? One possibility is that there are more opportunities for using, and therefore strengthening, knowledge about the equal sign than knowledge about the recognize-equivalence strategy. The relational interpretation of the equal sign is compatible with *all* instances of the equal sign that students encounter—even instances in which the operational interpretation also applies (e.g., $38 + 27 = \square$). Thus, once students begin to think about the equal sign relationally, they have many opportunities for that knowledge to be reinforced. Students are less likely to encounter opportunities to think about equivalent equations, at least until they begin to learn about algebraic manipulations—and even then, the instructional focus is often solely on performing the algebraic manipulations, without much explicit attention to understanding that the manipulations preserve the equivalence relation. Thus, there are comparatively fewer opportunities for the recognize-equivalence strategy to be reinforced, and students may need to invent it anew each of the first few times it is used.

Another likely possibility is that using the recognize-equivalence strategy requires realizing that knowledge of the equal sign can be applied to *efficiently* solve the problem, although other strategies can also lead to correct solutions. The equivalent equations items can be solved correctly using the solve-and-compare strategy and the substitution strategy, as well as the recognize-equivalence strategy. Correctness may be students' overarching consideration, so it is not surprising that students sometimes used these other correct, but less efficient, approaches. We consider the recognize-equivalence strategy to be the most sophisticated because of the deeper understanding of equality it reveals, as well as the problem-solving efficiency it provides.

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The present findings help build the case that it does matter *when* students acquire a sophisticated understanding of the equal sign. We found that students who showed this understanding earlier applied it more effectively in solving the equivalent equations items at the final time point. Moreover, this pattern held even when we controlled for later understanding by limiting the analysis to students who offered a relational definition at the final time point. It seems likely that students who achieve a relational understanding of the equal sign earlier may also have greater success on other tasks that involve the equal sign, such as solving algebraic equations. Kieran (1992) underscored this point, noting that a relational view is "helpful in successfully making the transition to the formal method of equation solving that involves performing the same operation on both sides of the equal sign" (p. 400). Thus, as students' knowledge of the relational meaning of the equal sign becomes stronger, it is more likely to be activated in contexts in which the equal sign appears, and consequently can guide students' solutions to novel problems that involve the equal sign.

Of course, it is also possible that some third variable might explain the association between early acquisition of a relational understanding of the equal sign, and later successful performance on the equivalent equations items. One candidate variable that could account for this relationship is mathematics ability. However, past research casts doubt on the idea that mathematics ability can fully explain the relationship between equal sign understanding and equation-solving performance. Knuth et al. (2006) examined the relationship between understanding of the equal sign and *concurrent* performance solving linear equations, and found that the two were associated, even when mathematics ability was statistically controlled.

In this study, we found that most students offered a relational definition of the equal sign before they used the recognize-equivalence strategy to solve the equivalent equations items. Thus, in most cases, conceptual knowledge about the equal sign appears to be acquired before the tendency to apply such knowledge in solving problems. Nevertheless, a substantial minority of students showed the reverse pattern—they used the recognize-equivalence strategy before they were able to provide an explicit relational definition for the equal sign. These findings are reminiscent of the debate about whether knowledge of concepts or procedures "comes first" in children's early counting, simple arithmetic, and other mathematical domains. Rittle-Johnson and Siegler (1999) argue that in domains in which children have many opportunities for practicing procedures, procedural knowledge appears to emerge first, whereas in domains in which children rarely have opportunities for practicing procedures, conceptual knowledge appears to emerge first. In the present study, the relevant question is not about concepts versus procedures, but about explicit conceptual knowledge versus the tendency to apply (perhaps implicit) conceptual knowledge in problem solving. Students rarely need to draw on relational knowledge of the equal sign in problem solving, at least until they reach algebra. Therefore, it makes sense that most students acquire explicit conceptual knowledge of the equal sign first, and only later use this knowledge to guide their problem solving. However, the reverse pattern may also occur, depending on individual students' experiences. For students who show the reverse pattern, recognizing equivalent equations may actually be a step on the path to inferring an explicit relational understanding of the equal sign. This might be especially likely to occur when students use the solve-and-compare strategy, find the solutions are the same, and then go back to examine the equations to figure out why.

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CONCLUDING REMARKS

The present findings highlight the need for continued attention to mathematical equivalence and, more specifically, the equal sign in middle school mathematics curriculum and instruction. Much of the difficulty students have in their early algebra learning, and in learning to solve algebraic equations, in particular, can be traced to a "compulsion to calculate" (Stacey & MacGregor, 1990, p. 151) developed during students' experiences in elementary school mathematics. With respect to the equal sign, this compulsion to calculate often fosters an operational view of the equal sign (Baroody & Ginsburg, 1983). Thus, students' success in early algebra learning may require providing them with opportunities to move beyond an operational view of the equal sign and toward a relational view—a view that in turn may facilitate the development of an understanding of the symmetric nature of an equation (i.e., the equivalence relationship between the expressions on the left- and right-hand sides of an equation).

As the findings underscore, however, just because a student displays a relational understanding at one point or in one particular problem context, does not imply that the student has a robust, flexible understanding of the concept. Students' understanding appears fragile at first, and students often fail to apply their understanding in all contexts in which it would be appropriate. Thus, continued attention to the equal sign in middle school mathematics curriculum and instruction should include varied and regular opportunities for students to develop a relational understanding. For example, building on students' arithmetic experiences, middle school teachers might present students with several number sentences and ask them to determine which are equivalent (e.g., $9 + 5 = 14$, $9 + 5 - 3 = 14 - 3$, and $9 + 5 - 3 = 14 + 3$), and then help students move beyond computation as their means of determination (i.e., help them to attend to the symmetry of the equations). As students progress through middle school, their opportunities to further develop a relational understanding of the equal sign should become more algebraic in nature (e.g., solving equivalent equations problems).

A sophisticated understanding of the equal sign is crucial for success in algebra. As Carpenter et al. (2003) have argued, a "limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation"

(p. 22). The present findings lend support to this claim, and they suggest that inculcating a sophisticated understanding of the equal sign at an earlier point in students' mathematical development may be most beneficial for their later success in algebra.

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