Elementary number theory in nine chapters

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The intriguing natural numbers

'The time has come,' the Walrus said, 'To talk of many things.'
Lewis Carroll

1.1 Polygonal numbers

We begin the study of elementary number theory by considering a few basic properties of the set of natural or counting numbers, $\{1, 2, 3, \ldots\}$. The natural numbers are closed under the binary operations of addition and multiplication. That is, the sum and product of two natural numbers are also natural numbers. In addition, the natural numbers are commutative, associative, and distributive under addition and multiplication. That is, for any natural numbers, a, b, c:

$$\begin{array}{lll} a + (b + c) = (a + b) + c, & a(bc) = (ab)c & \text{(associativity);} \\ a + b = b + a, & ab = ba & \text{(commutativity);} \\ a(b + c) = ab + ac, & (a + b)c = ac + bc & \text{(distributivity).} \end{array}$$

We use juxtaposition, xy, a convention introduced by the English mathematician Thomas Harriot in the early seventeenth century, to denote the product of the two numbers x and y. Harriot was also the first to employ the symbols '$>$' and '$<$' to represent, respectively, 'is greater than' and 'is less than'. He is one of the more interesting characters in the history of mathematics. Harriot traveled with Sir Walter Raleigh to North Carolina in 1585 and was imprisoned in 1605 with Raleigh in the Tower of London after the Gunpowder Plot. In 1609, he made telescopic observations and drawings of the Moon a month before Galileo sketched the lunar image in its various phases.

One of the earliest subsets of natural numbers recognized by ancient mathematicians was the set of polygonal numbers. Such numbers represent an ancient link between geometry and number theory. Their origin can be traced back to the Greeks, where properties of oblong, triangular, and square numbers were investigated and discussed by the sixth century BC, pre-Socratic philosopher Pythagoras of Samos and his followers. The

Greeks established the deductive method of reasoning whereby conclusions are derived using previously established results.

At age 18, Pythagoras won a prize for wrestling at the Olympic games. He studied with Thales, father of Greek mathematics, traveled extensively in Egypt and was well acquainted with Babylonian mathematics. At age 40, after teaching in Elis and Sparta, he migrated to Magna Graecia, where the Pythagorean School flourished at Croton in what is now Southern Italy. The Pythagoreans are best known for their theory of the transmigration of souls and their belief that numbers constitute the nature of all things. The Pythagoreans occupied much of their time with mysticism and numerology and were among the first to depict polygonal numbers as arrangements of points in regular geometric patterns. In practice, they probably used pebbles to illustrate the patterns and in doing so derived several fundamental properties of polygonal numbers. Unfortunately, it was their obsession with the deification of numbers and collusion with astrologers that later prompted Saint Augustine to equate mathematicans with those full of empty prophecies who would willfully sell their souls to the Devil to gain the advantage.

The most elementary class of polygonal numbers described by the early Pythagoreans was that of the oblong numbers. The nth oblong number, denoted by o_n, is given by $n(n+1)$ and represents the number of points in a rectangular array having n columns and $n+1$ rows. Since $2+4+\cdots+2n = 2(1+2+\cdots+n) = 2 \cdot n(n+1)/2 = n(n+1) = o_n$, the sum of the first n even numbers equals the nth oblong number. Diagrams for the first four oblong numbers, 2, 6, 12, and 20, are illustrated in Figure 1.1.

The triangular numbers, 1, 3, 6, 10, 15, ..., t_n, ..., where t_n denotes the nth triangular number, represent the numbers of points used to portray equilateral triangular patterns as shown in Figure 1.2. In general, from the sequence of dots in the rows of the triangles in Figure 1.2, it follows that t_n, for $n \geqslant 1$, represents successive partial sums of the first n natural numbers. For example, $t_4 = 1+2+3+4 = 10$. Since the natural numbers are commutative and associative,

$$t_n = 1+2+\cdots+(n-1)+n$$

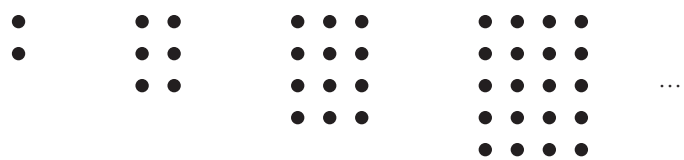

Figure 1.1

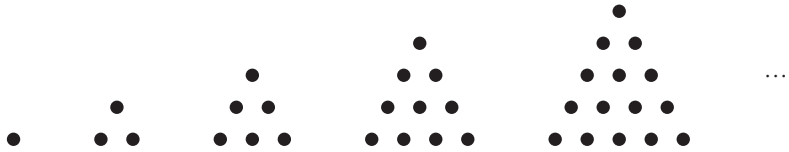

Figure 1.2

and

$$t_n = n + (n-1) + \cdots + 2 + 1;$$

adding columnwise, it follows that $2t_n = (n+1) + (n+1) + \cdots (n+1) = n(n+1)$. Hence, $t_n = n(n+1)/2$. Multiplying both sides of the latter equation by 2, we find that twice a triangular number is an oblong number. That is, $2t_n = o_n$, for any positive integer n. This result is illustrated in Figure 1.3 for the case when $n = 6$.

The square numbers, 1, 4, 9, 16, ..., were represented geometrically by the Pythagoreans as square arrays of points, as shown in Figure 1.4. In 1225, Leonardo of Pisa, more commonly known as Fibonacci, remarked, in *Liber quadratorum* (*The Book of Squares*) that the nth square number, denoted by s_n, exceeded its predecessor, s_{n-1}, by the sum of the two roots. That is $s_n = s_{n-1} + \sqrt{s_n} + \sqrt{s_{n-1}}$ or, equivalently, $n^2 = (n-1)^2 + n + (n-1)$. Fibonacci, later associated with the court of Frederick II, Emperor of the Holy Roman Empire, learned to calculate with Hindu–Arabic numerals while in Bougie, Algeria, where his father was a customs officer.

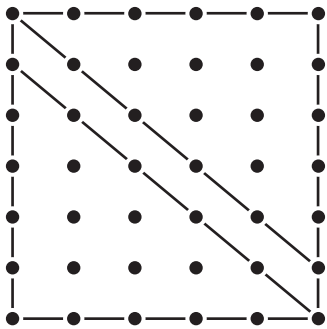

Figure 1.3

Figure 1.4

He was a direct successor to the Arabic mathematical school and his work helped popularize the Hindu–Arabic numeral system in Europe. The origin of Leonardo of Pisa's sobriquet is a mystery, but according to some sources, Leonardo was figlio de (son of) Bonacci and thus known to us patronymically as Fibonacci.

The Pythagoreans realized that the nth square number is the sum of the first n odd numbers. That is, $n^2 = 1 + 3 + 5 + \cdots + (2n - 1)$, for any positive integer n. This property of the natural numbers first appears in Europe in Fibonacci's *Liber quadratorum* and is illustrated in Figure 1.5, for the case when $n = 6$.

Another interesting property, known to the early Pythagoreans, appears in Plutarch's *Platonic Questions*. Plutarch, a second century Greek biographer of noble Greeks and Romans, states that eight times any triangular number plus one is square. Using modern notation, we have $8t_n + 1 = 8[n(n + 1)/2] + 1 = (2n + 1)^2 = s_{2n+1}$. In Figure 1.6, the result is illustrated for the case $n = 3$. It is in Plutarch's biography of Marcellus that we find one of the few accounts of the death of Archimedes during the siege of Syracuse, in 212 BC.

Around the second century BC, Hypsicles [HIP sih cleez], author of *Book XIV*, a supplement to Book XIII of Euclid's *Elements* on regular

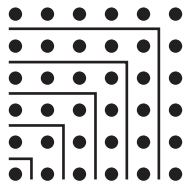

Figure 1.5

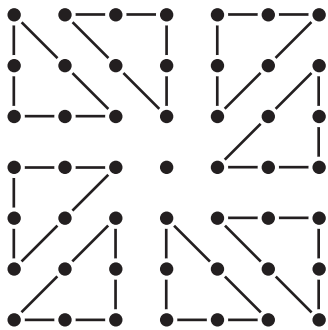

Figure 1.6

polyhedra, introduced the term polygonal number to denote those natural numbers that were oblong, triangular, square, and so forth. Earlier, the fourth century BC philosopher Plato, continuing the Pythagorean tradition, founded a school of philosophy near Athens in an area that had been dedicated to the mythical hero Academus. Plato's Academy was not primarily a place for instruction or research, but a center for inquiry, dialogue, and the pursuit of intellectual pleasure. Plato's writings contain numerous mathematical references and classification schemes for numbers. He firmly believed that a country's leaders should be well-grounded in Greek arithmetic, that is, in the abstract properties of numbers rather than in numerical calculations. Prominently displayed at the Academy was a maxim to the effect that none should enter (and presumably leave) the school ignorant of mathematics. The epigram appears on the logo of the American Mathematical Society. Plato's Academy lasted for nine centuries until, along with other pagan schools, it was closed by the Byzantine Emperor Justinian in 529.

Two significant number theoretic works survive from the early second century, *On Mathematical Matters Useful for Reading Plato* by Theon of Smyrna and *Introduction to Arithmetic* by Nicomachus [nih COM uh kus] of Gerasa. Smyrna in Asia Minor, now Izmir in Turkey, is located about 75 kilometers northeast of Samos. Gerasa, now Jerash in Jordan, is situated about 25 kilometers north of Amman. Both works are philosophical in nature and were written chiefly to clarify the mathematical principles found in Plato's works. In the process, both authors attempt to summarize the accumulated knowledge of Greek arithmetic and, as a consequence, neither work is very original. Both treatises contain numerous observations concerning polygonal numbers; however, each is devoid of any form of rigorous proofs as found in Euclid. Theon's goal was to describe the beauty of the interrelationships between mathematics, music, and astronomy. Theon's work contains more topics and was a far superior work mathematically than the *Introduction*, but it was not as popular. Both authors note that any square number is the sum of two consecutive triangular numbers, that is, in modern notation, $s_n = t_n + t_{n-1}$, for any natural number $n > 1$. Theon demonstrates the result geometrically by drawing a line just above and parallel to the main diagonal of a square array. For example, the case where $n = 5$ is illustrated in Figure 1.7. Nicomachus notes that if the square and oblong numbers are written alternately, as shown in Figure 1.8, and combined in pairs, the triangular numbers are produced. That is, using modern notation, $t_{2n} = s_n + o_n$ and $t_{2n+1} = s_{n+1} + o_n$, for any natural number n. From a standard multiplication table of the first ten natural

Table 1.1.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

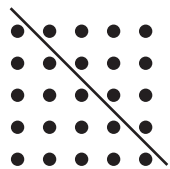

Figure 1.7

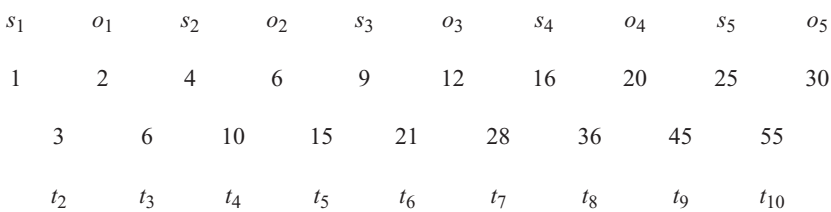

Figure 1.8

numbers, shown in Table 1.1, Nicomachus notices that the major diagonal is composed of the square numbers and the successive squares s_n and s_{n+1} are flanked by the oblong numbers o_n. From this, he deduces two properties that we express in modern notation as $s_n + s_{n+1} + 2o_n = s_{2n+1}$ and $o_{n-1} + o_n + 2s_n = s_{2n}$.

Nicomachus extends his discussion of square numbers to the higher dimensional cubic numbers, 1, 8, 27, 64, ..., and notes, but does not establish, a remarkable property of the odd natural numbers and the cubic numbers illustrated in the triangular array shown in Figure 1.9, namely, that the sum of the nth row of the array is n^3. It may well have been Nicomachus's only original contribution to mathematics.

1	1
3 5	8
7 9 11	27
13 15 17 19	64
21 23 25 27 29	125
.....	

Figure 1.9

In the *Introduction*, Nicomachus discusses properties of arithmetic, geometric, and harmonic progressions. With respect to the arithmetic progression of three natural numbers, he observes that the product of the extremes differs from the square of the mean by the square of the common difference. According to this property, known as the *Regula Nicomachi*, if the three terms in the progression are given by $a - k$, a, $a + k$, then $(a - k)(a + k) + k^2 = a^2$. In the Middle Ages, rules for multiplying two numbers were rather complex. The Rule of Nicomachus was useful in squaring numbers. For example, applying the rule for the case when $a = 98$, we obtain $98^2 = (98 - 2)(98 + 2) + 2^2 = 96 \cdot 100 + 4 = 9604$.

After listing several properties of oblong, triangular, and square numbers, Nicomachus and Theon discuss properties of pentagonal and hexagonal numbers. Pentagonal numbers, 1, 5, 12, 22, ..., ${p^5}_n$, ..., where ${p^5}_n$ denotes the nth pentagonal number, represent the number of points used to construct the regular geometric patterns shown in Figure 1.10. Nicomachus generalizes to heptagonal and octagonal numbers, and remarks on the patterns that arise from taking differences of successive triangular, square, pentagonal, heptagonal, and octagonal numbers. From this knowledge, a general formula for polygonal numbers can be derived. A practical technique for accomplishing this involving successive differences appeared in a late thirteenth century Chinese text *Works and Days Calendar* by Wang Xun and Guo Shoujing. The method was mentioned in greater detail in 1302 in *Precious Mirror of the Four Elements* by Zhu Shijie, a wandering

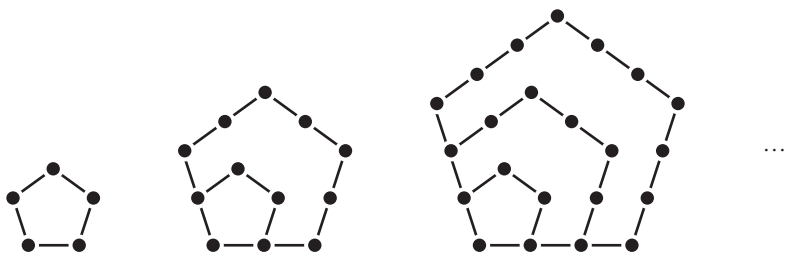

Figure 1.10

scholar who earned his living teaching mathematics. The method of finite differences was rediscovered independently in the seventeenth century by the British mathematicians Thomas Harriot, James Gregory, and Isaac Newton.

Given a sequence, a_k, a_{k+1}, a_{k+2}, ..., of natural numbers whose rth differences are constant, the method yields a polynomial of degree $r-1$ representing the general term of the given sequence. Consider the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{for } 0 \leqslant k \leqslant n, \quad \binom{n}{0} = 1, \quad \text{and otherwise } \binom{n}{k} = 0,$$

where for any natural number n, n factorial, written $n!$, represents the product $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ and, for consistency, $0! = 1$. The exclamation point used to represent factorials was introduced by Christian Kramp in 1802. The numbers, $\binom{n}{k}$, are called the binomial coefficients because of the role they play in the expansion of $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$. For example,

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

Denote the ith differences, Δ_i, of the sequence a_k, a_{k+1}, a_{k+2}, ... by $d_{i1}, d_{i2}, d_{i3}, \ldots$, and generate the following finite difference array:

n	k	$k+1$	$k+2$	$k+3$	$k+4$	$k+5$	$k+6$
a_n	a_k	a_{k+1}	a_{k+2}	a_{k+3}	a_{k+4}	a_{k+5}	a_{k+6}
Δ_1	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	
Δ_2		d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	
...							
Δ_r			d_{r1}	d_{r2}	d_{r3}	d_{r4}	

If the rth differences $d_{r1}, d_{r2}, d_{r3}, \ldots$ are equal, then working backwards and using terms in the leading diagonal each term of the sequence a_k, a_{k+1}, a_{k+2}, ... can be determined. More precisely, the finite difference array for the sequence $b_n = \binom{n-k}{m}$, for $m = 0, 1, 2, 3, \ldots, r$, $n = k, k+1, k+2, \ldots$, and a fixed value of k, has the property that the mth differences, Δ_m, consist of all ones and, except for $d_{m1} = 1$ for $1 \leqslant m \leqslant r$, the leading diagonal is all zeros. For example, if $m = 0$, the finite difference array for $a_n = \binom{n-k}{0}$ is given by

n	k	$k+1$	$k+2$	$k+3$	$k+4$	$k+5$	$k+6$
b_n	1	1	1	1	1	1	1
Δ_1	0	0	0	0	0	0	

If $m = 1$, the finite difference array for $a_n = \binom{n-k}{1}$ is given by

n	k	$k+1$	$k+2$	$k+3$	$k+4$	$k+5$	$k+6$
b_n	0	1	2	3	4	5	6
Δ_1	1	1	1	1	1	1	
Δ_2		0	0	0	0	0	0

If $m = 2$, the finite difference array for $a_n = \binom{n-k}{2}$ is given by

n	k	$k+1$	$k+2$	$k+3$	$k+4$	$k+5$	$k+6$
b_n	0	0	1	3	6	10	15
Δ_1	0	1	2	3	4	5	
Δ_2		1	1	1	1	1	1
Δ_3		0	0	0	0	0	

The leading diagonals of the finite difference array for the sequence a_k, a_{k+1}, a_{k+2}, ..., and the array defined by

$$a_k\binom{n-k}{0} + d_{11}\binom{n-k}{1} + d_{21}\binom{n-k}{2} + \cdots + d_{r1}\binom{n-k}{r}$$

are identical. Therefore,

$$a_n = a_k\binom{n-k}{0} + d_{11}\binom{n-k}{1} + d_{21}\binom{n-k}{2} + \cdots + d_{r1}\binom{n-k}{r},$$

for $n = k, k+1, k+2, \ldots$.

Example 1.1 The finite difference array for the pentagonal numbers, 1, 5, 12, 22, 35, ..., ${p^5}_n$, ... is given by

n	1	2	3	4	5	6	...
${p^5}_n$	1	5	12	22	35	51	...
Δ_1	4	7	10	13	16	...	
Δ_2		3	3	3	3	...	

Our indexing begins with $k = 1$. Therefore

$${p^5}_n = 1 \cdot \binom{n-1}{0} + 4 \cdot \binom{n-1}{1} + 3 \cdot \binom{n-1}{2} = 1 + 4(n-1) + 3\frac{(n-1)(n-2)}{2}$$

$$= \frac{3n^2 - n}{2}.$$

From Table 1.2, Nicomachus infers that the sum of the nth square and the $(n-1)$st triangular number equals the nth pentagonal number, that is, for any positive integer n, ${p^5}_n = s_n + t_{n-1}$. For example, if $n = 6$, $s_6 + t_5 = 36 + 15 = 51 = {p^5}_6$. He also deduces from Table 1.2 that three times the $(n-1)$st triangular number plus n equals the nth pentagonal number. For example, for $n = 9$, $3 \cdot t_8 + 9 = 3 \cdot 36 + 9 = 117 = {p^5}_9$.

In general, the m-gonal numbers, for $m = 3, 4, 5, \ldots$, where m refers to the number of sides or angles of the polygon in question, are given by

Table 1.2.

n	1	2	3	4	5	6	7	8	9	10
Triangular	1	3	6	10	15	21	28	36	45	55
Square	1	4	9	16	25	36	49	64	81	100
Pentagonal	1	5	12	22	35	51	70	92	117	145
Hexagonal	1	6	15	28	45	66	91	120	153	190
Heptagonal	1	7	18	34	55	81	112	148	189	235
Octagonal	1	8	21	40	65	96	133	176	225	280
Enneagonal	1	9	24	46	75	111	154	204	261	325
Decagonal	1	10	27	52	85	126	175	232	297	370

the sequence of numbers whose first two terms are 1 and m and whose second common differences equal $m-2$. Using the finite difference method outlined previously we find that ${p^m}_n = (m-2)n^2/2 - (m-4)n/2$, where ${p^m}_n$ denotes the nth m-gonal number. Triangular numbers correspond to 3-gonal numbers, squares to 4-gonal numbers, and so forth. Using Table 1.2, Nicomachus generalizes one of his previous observations and claims that ${p^m}_n + {p^3}_{n-1} = {p^{m+1}}_n$, where ${p^3}_n$ represents the nth triangular number.

The first translation of the *Introduction* into Latin was done by Apuleius of Madaura shortly after Nicomachus's death, but it did not survive. However, there were a number of commentaries written on the *Introduction*. The most influential, *On Nicomachus's Introduction to Arithmetic*, was written by the fourth century mystic philosopher Iamblichus of Chalcis in Syria. The Islamic world learned of Nicomachus through Thabit ibn Qurra's *Extracts from the Two Books of Nicomachus*. Thabit, a ninth century mathematician, physician, and philosopher, worked at the House of Wisdom in Baghdad and devised an ingenious method to find amicable numbers that we discuss in Chapter 4. A version of the *Introduction* was written by Boethius [beau EE thee us], a Roman philosopher and statesman who was imprisoned by Theodoric King of the Ostrogoths on a charge of conspiracy and put to death in 524. It would be hard to overestimate the influence of Boethius on the cultured and scientific medieval mind. His *De institutione arithmetica libri duo* was the chief source of elementary mathematics taught in schools and universities for over a thousand years. He coined the term *quadrivium* referring to the disciplines of arithmetic, geometry, music, and astronomy. These subjects together with the *trivium* of rhetoric, grammar, and logic formed the seven liberal arts popularized in the fifth century in Martianus Capella's book *The Marriage of Mercury*

and Philology. Boethius's edition of Nicomachus's *Introduction* was the main medium through which the Romans and people of the Middle Ages learned of formal Greek arithmetic, as opposed to the computational arithmetic popularized in the thirteenth and fourteenth centuries with the introduction of Hindu–Arabic numerals. Boethius wrote *The Consolation of Philosophy* while in prison where he reflected on the past and on his outlook on life in general. The *Consolation* was translated from Latin into Anglo-Saxon by Alfred the Great and into English by Chaucer and Elizabeth I.

In the fourth century BC Philip of Opus and Speusippus wrote treatises on polygonal numbers that did not survive. They were, however, among the first to extend polygonal numbers to pyramidal numbers. Speusippus [spew SIP us], a nephew of Plato, succeeded his uncle as head of the Academy. Philip, a mathematician–astronomer, investigated the connection between the rainbow and refraction. His native home Opus, the modern town of Atalandi, on the Euboean Gulf, was a capital of one of the regions of Locris in Ancient Greece.

Each class of pyramidal number is formed from successive partial sums of a specific type of polygonal number. For example, the nth tetrahedral number, $P^3{}_n$, can be obtained from successive partial sums of triangular numbers, that is, $P^3{}_n = p^3{}_1 + p^3{}_2 + \cdots + p^3{}_n$. For example, $P^3{}_4 = 1 + 3 + 6 + 10 = 20$. Accordingly, the first four tetrahedral numbers are 1, 4, 10, and 20. An Egyptian papyrus written about 300 BC gives $\frac{1}{2}(n^2 + n)$ as the sum of the first n natural numbers and $\frac{1}{3}(n + 2)\frac{1}{2}(n^2 + n)$ as the sum of the first n triangular numbers. That is, $t_n = p^3{}_n = n(n + 1)/2$ and $P^3{}_n = n(n + 1)(n + 2)/6$. The formula for $P^3{}_n$ was derived by the sixth century Indian mathematician–astronomer Aryabhata who calculated one of the earliest tables of trigonometric sines using 3.146 as an estimate for π.

Example 1.2 Each triangle on the left hand side of the equality in Figure 1.11 gives a different representation of the first four triangular numbers, 1, 3 $(1 + 2)$, 6 $(1 + 2 + 3)$, and 10 $(1 + 2 + 3 + 4)$. Hence, $3 \cdot (1 + 3 + 6 + 10) = 1 \cdot 6 + 2 \cdot 6 + 3 \cdot 6 + 4 \cdot 6 = (1 + 2 + 3 + 4) \cdot 6 = t_4(4 + 2)$. In

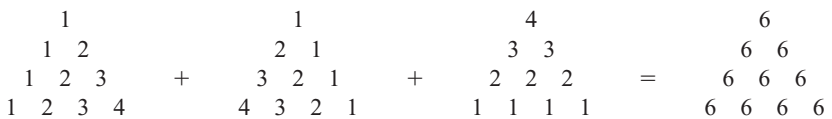

Figure 1.11

Table 1.3.

	$p^3{}_1$	$p^3{}_2$	$p^3{}_3$	$p^3{}_4$	$p^3{}_5$	$p^3{}_6$	$p^3{}_7$	$p^3{}_8$	$p^3{}_9$
	1	4	10	20	35	56	84	120	165
1	2	3	4	5	6	7	8	9	
2	4	6	8	10	12	14	16		
3	6	9	12	15	18	21			
4	8	12	16	20	24				
5	10	15	20	25					
6	12	18	24						
7	14	21							
8	16								
9									

general, $3(t_1 + t_2 + t_3 + \cdots + t_n) = t_n(n+2) = n(n+1)(n+2)/2$. Therefore, $P^3{}_n = n(n+1)(n+2)/6$.

In Figure 1.11, the sum of the numbers in the third triangle is the fourth tetrahedral number. That is, $1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 20$. Thus, in general, $1 \cdot n + 2 \cdot (n-1) + \cdots + (n-1) \cdot 2 + n \cdot 1 = P^3{}_n$. Hence, we can generate the tetrahedral numbers by summing the terms in the SW–NE diagonals of a standard multiplication table as shown in Table 1.3. For example, $P^3{}_6 = 6 + 10 + 12 + 12 + 10 + 6 = 56$.

Pyramidal numbers with a square base are generated by successive partial sums of square numbers. Hence, the nth pyramidal number, denoted by $P^4{}_n$, is given by $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$. For example, $P^4{}_4 = 1 + 4 + 9 + 16 = 30$. The total number of cannonballs in a natural stacking with a square base is a pyramidal number.

Slicing a pyramid through a vertex and the diagonal of the opposite base results in two tetrahedrons. Hence, it should be no surprise to find that the sum of two consecutive tetrahedral numbers is a pyramidal number, that is, $P^4{}_n = P^3{}_{n-1} + P^3{}_n$.

In the tenth century, Gerbert of Aurillac in Auvergne included a number of identities concerning polygonal and pyramidal numbers in his correspondence with his pupil Adalbold, Bishop of Utrecht. Much of Gerbert's *Geometry* was gleaned from the work of Boethius. One of the more difficult problems in the book asks the reader to find the legs of a right triangle given the length of its hypotenuse and its area. Gerbert was one of the first to teach the use of Hindu–Arabic numerals and promoted the utilization of zero as a digit. He was elected Pope Sylvester II in 999, but his reign was short.

Table 1.4.

n	1	2	3	4	5	6	7	8	9	10
${f^0}_n$	1	1	1	1	1	1	1	1	1	1
${f^1}_n$	1	2	3	4	5	6	7	8	9	10
${f^2}_n$	1	3	6	10	15	21	28	36	45	55
${f^3}_n$	1	4	10	20	35	56	84	120	165	220
${f^4}_n$	1	5	15	35	70	126	210	330	495	715
${f^5}_n$	1	6	21	56	126	252	462	792	1287	2002
${f^6}_n$	1	7	28	84	210	462	924	1716	3003	5005

Triangular and tetrahedral numbers form a subclass of the figurate numbers. In the 1544 edition of *Arithmetica Integra*, Michael Stifel defined the nth mth-order figurate number, denoted by ${f^m}_n$, as follows: ${f^m}_n = {f^m}_{n-1} + {f^{m-1}}_n$, ${f^m}_1 = {f^0}_n = {f^0}_1 = 1$, for $n = 2, 3, \ldots$, and $m = 1, 2, 3, \ldots$. An array of figurate numbers is illustrated in Table 1.4, where the nth triangular number corresponds to ${f^2}_n$ and the nth tetrahedral number to ${f^3}_n$. In 1656, John Wallis, the English mathematician who served as a cryptanalyst for several Kings and Queens of England, and introduced the symbol ∞ to represent infinity, showed that, for positive integers n and r, ${f^r}_{n+1} = {f^0}_n + {f^1}_n + {f^2}_n + \cdots + {f^r}_n$.

Stifel was the first to realize a connection existed between figurate numbers and binomial coefficients, namely that ${f^m}_n = \binom{n+m-1}{m}$. In particular, ${f^2}_n = t_n = \binom{n+1}{2}$ and ${f^3}_n = {P^3}_n = \binom{n+2}{3}$. Stifel earned a Master's degree at Wittenberg University. He was an avid follower of Martin Luther, an ardent biblical scholar, and a millenarian. Stifel must have thought he was standing in the foothills of immortality when, through his reading, he inferred that the world was going to end at 8 o'clock on the morning of October 18, 1533. He led a band of followers to the top of a nearby hill to witness the event, a nonoccurrence that did little to enhance his credibility.

Nicomachus's *Introduction to Arithmetic* was one of the most significant ancient works on number theory. However, besides Books VII–IX of Euclid's *Elements*, whose contents we will discuss in the next chapter, the most influential number theoretic work of ancient times was the *Arithmetica* of Diophantus, one of the oldest algebra treatises in existence. Diophantus, a mathematician who made good use of Babylonian and Greek sources, discussed properties of polygonal numbers and included a rule to determine the nth m-gonal number which he attributed to Hypsicles. Unfortunately, a complete copy of the *Arithmetica* was lost when the Library of Alexandria was vandalized in 391 by Christians acting under the

aegis of Theophilus, Bishop of Alexandria, and a decree by Emperor Theodosius concerning pagan monuments. Portions of the treatise were rediscovered in the fifteenth century. As a consequence, the *Arithmetica* was one of the last Greek mathematical works to be translated into Latin.

There were a number of women who were Pythagoreans, but Hypatia, the daughter of the mathematician Theon of Alexandria, was the only notable female scholar in the ancient scientific world. She was one of the last representatives of the Neo-platonic School at Alexandria, where she taught science, art, philosophy, and mathematics in the early fifth century. She wrote a commentary, now lost, on the first six books of the *Arithmetica* and may very well have been responsible for editing the version of Ptolemy's *Almagest* that has survived. Some knowledge of her can be gleaned from the correspondence between her and her student Synesius, Bishop of Cyrene. As a result of her friendship with Alexandria's pagan Prefect, Orestes, she incurred the wrath of Cyril, Theophilus's nephew who succeeded him in 412 as Bishop of Alexandria. In 415, Hypatia was murdered by a mob of Cyril's followers. During the millennium following her death no woman distinguished herself in science or mathematics.

In the introduction to the *Arithmetica*, Diophantus refers to his work as consisting of thirteen books, where a book consisted of a single scroll representing material covered in approximately twenty to fifty pages of ordinary type. The first six books of the *Arithmetica* survived in Greek and four books, which may have a Hypatian rather than a Diophantine origin, survived in Arabic. In addition, a fragment on polygonal numbers by Diophantus survives in Greek. The *Arithmetica* was not a textbook, but an innovative handbook involving computations necessary to solve practical problems. The *Arithmetica* was the first book to introduce consistent algebraic notation and systematically use algebraic procedures to solve equations. Diophantus employed symbols for squares and cubes but limited himself to expressing each unknown quantity in terms of a single variable. Diophantus is one the most intriguing and least known characters in the history of mathematics.

Much of the *Arithmetica* consists of cleverly constructed positive rational solutions to more than 185 problems in indeterminate analysis. Negative solutions were not acceptable in Diophantus's time or for the next 1500 years. By a rational solution, we mean a number of the form p/q, where p and q are integers and $q \neq 0$. In one example, Diophantus constructed three rational numbers with the property that the product of any two of the numbers added to their sum or added to the remaining number is square. That is, in modern notation, he determined numbers x, y,

z such that $xy + x + y$, $xz + x + z$, $yz + y + z$, $xy + z$, $xz + y$, and $yz + x$ are all square. In another problem, Diophantus found right triangles with sides of rational length such that the length of the hypotenuse minus the length of either side is a cube. In the eleventh century, in Baghdad, the Islamic mathematician al-Karaji and his followers expanded on the methods of Diophantus and in doing so undertook a systematic study of the algebra of exponents.

Problems similar to those found in the *Arithmetica* first appear in Europe in 1202 in Fibonacci's *Liber abaci* (*Book of Calculations*). The book introduced Hindu–Arabic numerals to European readers. It was revised by the author in 1228 and first printed in 1857. However, the first reference to Diophantus's works in Europe is found in a work by Johannes Müller who, in his day, was called Joannes de Regio monte (John of Königsberg). However, Müller is perhaps best known today by his Latinized name Regiomontanus, which was popularized long after his death. Regiomontanus, the first publisher of mathematical and astronomical literature, studied under the astronomer Georges Peurbach at the University of Vienna. He wrote a book on triangles and finished Peurbach's translation of Ptolemy's *Almagest*. Both Christopher Columbus and Amerigo Vespucci used his *Ephemerides* on their voyages. Columbus, facing starvation in Jamaica, used a total eclipse of the Moon on February 29, 1504, predicted in the *Ephemerides*, to encourage the natives to supply him and his men with food. A similar idea, albeit using a total solar eclipse, was incorporated by Samuel Clemens (Mark Twain) in *A Connecticut Yankee in King Arthur's Court*. Regiomontanus built a mechanical fly and a 'flying' eagle, regarded as the marvel of the age, which could flap its wings and saluted when Emperor Maximilian I visited Nuremberg. Domenico Novarra, Copernicus's teacher at Bologna, regarded himself as a pupil of Regiomontanus who, for a short period, lectured at Padua.

Regiomontanus wrote to the Italian mathematician Giovanni Bianchini in February 1464 that while in Venice he had discovered Greek manuscripts containing the first six books of *Arithmetica*. In 1471, Regiomontanus was summoned to Rome by Pope Sixtus IV to reform the ecclesiastical calendar. However, in 1476, before he could complete his mission, he died either a victim of the plague or poisoned for his harsh criticism of a mediocre translation of the *Almagest*.

In 1572, an Italian engineer and architect, Rafael Bombelli, published *Algebra*, a book containing the first description and use of complex numbers. The book included 271 problems in indeterminate analysis, 147 of which were borrowed from the first four books of Diophantus's

Arithmetica. Gottfried Leibniz used Bombelli's text as a guide in his study of cubic equations. In 1573, based on manuscripts found in the Vatican Library, Wilhelm Holtzman, who wrote under the name Xylander, published the first complete Latin translation of the first six books of the *Arithmetica*. The Dutch mathematician, Simon Stevin, who introduced a decimal notation to European readers, published a French translation of the first four books of the *Arithmetica*, based on Xylander's work.

In 1593, François Viète, a lawyer and cryptanalyst at the Court of Henry IV, published *Introduction to the Analytic Art*, one of the first texts to champion the use of Latin letters to represent numbers to solve problems algebraically. In an effort to show the power of algebra, Viète included algebraic solutions to a number of interesting problems that were mentioned but not solved by Diophantus in the *Arithmetica*.

A first-rate translation, *Diophanti Alexandrini arithmeticorum libri sex*, by Claude-Gaspard Bachet de Méziriac, appeared in 1621. Bachet, a French mathematician, theologian, and mythologist of independent means, included a detailed commentary with his work. Among the number theoretic results Bachet established were

(a) $p^m{}_{n+r} = p^m{}_n + p^m{}_r + nr(m-2)$,
(b) $p^m{}_n = p^3{}_n + (m-3)p^3{}_{n-1}$, and
(c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (p^3{}_n)^2$,

where $p^m{}_n$ denotes the nth m-gonal number. The third result is usually expressed as $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ and referred to as Lagrange's identity.

In the fourth book of the *Arithmetica* Diophantus found three rational numbers, $\frac{153}{81}$, $\frac{6400}{81}$, and $\frac{8}{81}$, which if multiplied in turn by their sum yield a triangular number, a square number, and a cube, respectively. Bachet extended the problem to one of finding five numbers which if multiplied in turn by their sum yield a triangular number, a square, a cube, a pentagonal number, and a fourth power, respectively.

Bachet was an early contributor to the field of recreational mathematics. His *Problèmes plaisants et délectables qui se font par les nombres*, first published in 1612, is replete with intriguing problems including a precursor to the cannibals and missionaries problem, the Christians and Turks problem, and a discussion on how to create magic squares. At age 40, Bachet married, retired to his country estate, sired seven children, and gave up his mathematical activity forever. Except for recurring bouts with gout and rheumatism, he lived happily ever after.

The rediscovery of Diophantus's work, in particular through Bachet's

edition which relied heavily on Bombelli's and Xylander's work, greatly aided the renaissance of mathematics in Western Europe. One of the greatest contributors to that renaissance was Pierre de Fermat [fair MAH], a lawyer by profession who served as a royal councillor at the Chamber of Petitions at the Parlement of Toulouse. Fermat was an outstanding amateur mathematician. He had a first-class mathematical mind and, before Newton was born, discovered a method for finding maxima and minima and general power rules for integration and differentiation of polynomial functions of one variable. He rarely, however, published any of his results. In 1636, he wrote, in a burst of enthusiasm, that he had just discovered the very beautiful theorem that every positive integer is the sum of at most three triangular numbers, every positive integer is the sum of at most four squares, every positive integer is the sum of at most five pentagonal numbers, and so on *ad infinitum*, but added, however, that he could not give the proof, since it depended on 'numerous and abstruse mysteries of numbers'. Fermat planned to devote an entire book to these mysteries and to 'effect in this part of arithmetic astonishing advances over the previously known limits'. Unfortunately, he never published such a book.

In 1798, in *Théorie des nombres*, the Italian mathematician and astronomer, Joseph-Louis Lagrange, used an identity discovered by the Swiss mathematician Leonhard Euler to prove Fermat's claim for the case of square numbers. Karl Friedrich Gauss proved the result for triangular numbers when he was nineteen and wrote in his mathematical diary for 10 July 1796: '$\varepsilon\upsilon\rho\eta\kappa\alpha$! num $= \blacktriangle + \blacktriangle + \blacktriangle$.' Two years later, Gauss's result was proved independently by the French mathematician, Adrien Marie Legendre. In the introduction to *Disquisitiones arithmeticae* (*Arithmetical Investigations*) Gauss explains his indebtedness to Diophantus's *Arithmetica*. In Chapters 5, 6, and 8, we discuss the contents of Gauss's *Disquisitiones*. In 1808, Legendre included a number of quite remarkable number theoretic results in his *Théorie des nombres*; in particular, the property that every odd number not of the form $8k + 7$, where k is a positive integer, can be expressed as the sum of three or fewer square numbers. In 1815, Augustin-Louis Cauchy proved that every positive integer is the sum of m m-gonal numbers of which all but four are equal to 0 or 1. Cauchy's *Cours d'analyse*, published in 1821, advocated a rigorous approach to mathematical analysis, in particular to the calculus. Unfortunately, Cauchy was very careless with his correspondence. Evariste Galois and Niels Henrik Abel sent brilliant manuscripts to Cauchy for his examination and evaluation, but they were lost.

One of the first results Fermat established was that nine times any

triangular number plus one always yielded another triangular number. Fermat later showed that no triangular number greater than 1 could be a cube or a fourth power. Fermat, always the avid number theorist, once challenged Lord Brouncker, first President of the Royal Society, and John Wallis, the best mathematician in England at the time, to prove there is no triangular number other than unity that is a cube or a fourth power. Neither was able to answer his query.

Fermat often used the margins of texts to record his latest discoveries. In 1670, Fermat's son, Clément-Samuel, published a reprint of Bachet's *Diophantus* together with his father's marginal notes and an essay by the Jesuit, Jacques de Billy, on Fermat's methods for solving certain types of Diophantine-type equations. His most famous marginal note, the statement of his 'last' theorem, appears in his copy of Bachet's edition of the *Arithmetica*. Fermat wrote to the effect that it was impossible to separate a cube into two cubes, or a biquadratic into two biquadratics, or generally any power except a square into two powers with the same exponent. Fermat added that he had discovered a truly marvelous proof of this result; however, the margin was not large enough to contain it. Fermat's Last Theorem was 'last' in the sense that it was the last major conjecture by Fermat that remained unproven. Fermat's Last Theorem has proven to be a veritable fountainhead of mathematical research and until recently its proof eluded the greatest mathematicians. In 'The Devil and Simon Flagg' Arthur Porges relates a delightful tale in which the Devil attempts to prove Fermat's Last Theorem.

The Swiss mathematician, Leonhard Euler [oiler], perused a copy of Bachet's Diophantus with Fermat's notes and was intrigued by Fermat's emphasis on integer, rather than rational, solutions. At the University of Basel, Euler was a student of Johann Bernoulli. Bernoulli won the mathematical prize offered by the Paris Academy twice. His son Daniel Bernoulli won it ten times. Euler, who won the prize twelve times, began a lifelong study of number theory at age 18. Euler's papers are remarkably readable. He has a good historical sense and often informs the reader of things that have impressed him and of ideas that led him to his discoveries. Even though over half of Euler's 866 publications were written when he was blind, he laid the foundation of the theory of numbers as a valid branch of mathematics. His works were still appearing in the *Memoirs* of the St Petersburg Academy fifty years after his death. It is estimated that he was responsible for one-third of all the mathematical work published in Europe from 1726 to 1800. He had a phenomenal memory and knew Vergil's *Aeneid* by heart. At age 70, given any page number from the edition he

owned as a youth, he could recall the top and bottom lines. In addition, he kept a table of the first six powers of the first hundred positive integers in his head.

Before proceeding further, it is important in what follows for the reader to be able to distinguish between a conjecture and an open question. By a conjecture we mean a statement which is thought to be true by many, but has not been proven yet. By an open question we mean a statement for which the evidence is not very convincing one way or the other. For example, it was conjectured for many years that Fermat's Last Theorem was true. It is an open question, however, whether $4! + 1 = 5^2$, $5! + 1 = 11^2$, and $7! + 1 = 71^2$ are the only squares of the form $n! + 1$.

Exercises 1.1

1. An even number can be expressed as $2n$ and an odd number as $2n + 1$, where n is a natural number. Two natural numbers are said to be of the same parity if they are either both even or both odd, otherwise they are said to be of opposite parity. Given any two natural numbers of the same parity, show that their sum and difference are even. Given two numbers of opposite parity, show that their sum and difference are odd.
2. Nicomachus generalized oblong numbers to rectangular numbers, which are numbers of the form $n(n + k)$, denoted by $r_{n,k}$, where $k \geqslant 1$ and $n > 1$. Determine the first ten rectangular numbers that are not oblong.
3. Prove algebraically that the sum of two consecutive triangular numbers is always a square number.
4. Show that $9t_n + 1$ [Fermat], $25t_n + 3$ [Euler], and $49t_n + 6$ [Euler] are triangular.
5. Show that the difference between the squares of any two consecutive triangular numbers is always a cube.
6. In 1991, S.P. Mohanty showed that there are exactly six triangular numbers that are the product of three consecutive integers. For example, $t_{20} = 210 = 5 \cdot 6 \cdot 7$. Show that t_{608} is the product of three consecutive positive integers.
7. Show that the product of any four consecutive natural numbers plus one is square. That is, show that for any natural number n, $n(n + 1)(n + 2)(n + 3) + 1 = k^2$, for some natural number k.
8. The nth star number, denoted by $*_n$, represents the sum of the nth square number and four times the $(n - 1)$st triangular number, where

$*_1 = 1$. One geometric interpretation of star numbers is as points arranged in a square with equilateral triangles on each side. For example $*_2$ is illustrated in Figure 1.12. Derive a general formula for the nth star number.

9. Show that Gauss's discovery that every number is the sum of three or fewer triangular numbers implies that every number of the form $8k + 3$ can be expressed as the sum of three odd squares.
10. Verify Nicomachus's claim that the sum of the odd numbers on any row in Figure 1.9 is a cube.
11. For any natural number n prove that
    (a) $s_{2n+1} = s_n + s_{n+1} + 2o_n$. [Nicomachus]
    (b) $s_{2n} = o_{n-1} + o_n + 2s_n$. [Nicomachus]
12. Show that $s_n + t_{n-1} = {p^5}_n$, for any natural number n. [Nicomachus]
13. Prove that ${p^5}_n = 3t_{n-1} + n$, for any natural number n. [Nicomachus]
14. Show that every pentagonal number is one-third of a triangular number.
15. Find a positive integer $n > 1$ such that $1^2 + 2^2 + 3^2 + \cdots + n^2$ is a square number. [*Ladies' Diary*, 1792] This problem was posed by Edouard Lucas in 1875 in *Annales de Mathématique Nouvelles*. In 1918, G. N. Watson proved that the problem has a unique solution.
16. Prove the square of an odd multiple of 3 is the difference of two triangular numbers, in particular show that for any natural number n, $[3(2n + 1)]^2 = t_{9n+4} - t_{3n+1}$.
17. Show that there are an infinite number of triangular numbers that are the sum of two triangular numbers by establishing the identity $t_{[n(n+3)+1]/2} = t_{n+1} + t_{n(n+3)/2}$.
18. Prove that $t_{2mn+m} = 4m^2 t_n + t_m + mn$, for any positive integers m and n.
19. Paul Haggard and Bonnie Sadler define the nth m-triangular number, ${T^m}_n$, by ${T^m}_n = n(n + 1) \cdots (n + m + 1)/(m + 2)$. When $m = 0$, we obtain the triangular numbers. Generate the first ten ${T^1}_n$ numbers.

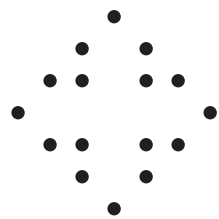

Figure 1.12